PRINCETON UNIVERSITY

DEPARTMENT OF AERONAUTICAL ENGINEERING

ON THE STABILITY OF AIRBORNE TOWED VEHICLES

BY

MICHEAL B. BURTON AND HARRY PASSMORE III

REPORT NO. 603 MAY 1962

U. S. NAVAL AIR DEVELOPMENT CENTER
JOHNSVILLE, PENNSYLVANIA
CONTRACT N62269-1369

BIBLIOGRAPHICAL CONTROL SHEET

1. Originating agency and sponsor:

   O.A.: Princeton University, Princeton, New Jersey.
   S. : U. S. Naval Air Development Center, Johnsville, Pennsylvania.

2. Originating agency report number: Aeronautical Engineering Department Report No. 603.

3. Title and classification of title: On the Stability of Airborne Towed Vehicles. (Unclassified)

4. Personal authors: Burton, Micheal B. and Harry Passmore III

5. Date of report: May 1962

6. Pages: 74

7. Illustrative material: 27 figures

8. Prepared for Contract No.: N62269-1369

9. Prepared for Wep Task No.: RUDC-4B000
   Problem Assignment No. 303

10. Security classification: Unclassified

11. Distribution limitations: 6 copies, NADC, Johnsville

12. Abstract:

Studies of the dynamic stability of heavy bodies with lifting surfaces towed behind an airplane are presented. Following a fairly complete analysis of cable dynamics, analyses of the longitudinal, lateral and coupled motions of the system are developed and discussed. A current configuration is chosen as a starting point from which the studies are expanded to develop stability boundaries of the important parameters.

TABLE OF CONTENTS

ERRATA

Page	Line	Correction
Title	6 down	JOHNSVILLE
10	9 down	$\frac{\partial z}{\partial t} = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi x}{\ell} \left[-\frac{n\pi}{\ell} \quad \sin \frac{n\pi\lambda}{\ell} + \right]$
22	Equa. II-9	$T = T_o + wz$
2	Equa. II-	Last coefficient should be: $\sigma = \frac{s}{\frac{T_o}{K}}$
2	Equa. II-99	$(1+\mu\eta)\sin\lambda\,\partial\lambda/\partial\eta = \cos^2\lambda + \mu\cos\lambda - 1$
2	Equa. II-100	$\int_0^{\eta} \frac{\mu\,\partial\eta}{1+\mu\eta} = \int_0^{\pi/2} \frac{\mu \sin\lambda\,\partial\lambda}{1-\mu\cos\lambda-\cos^2\lambda}$
2.	17 down	Equation II-28 .
2	21 down	$\frac{\partial^2 z}{\partial x^2} = \partial z/\partial x = \partial^2 x/\partial t^2 = \partial z/\partial t = 0$
37	Equa. III-3	$\bar{F} = m\,\partial\bar{V}/\partial t + m(\bar{\omega} \times \bar{V})$
38	Equa. III-8	$r = $ [illegible]
43	6 up	$d(T\cos\lambda)/dz$
43	4 up	$\frac{d(T\cos\lambda)}{dz} = -T_o \sin\lambda_o \frac{d\lambda}{dz} + \cos\lambda \frac{dT}{dz}$
48	Equa. IV-2	$\begin{matrix} (Z_q + U_o)s & -U_o s + (Z_\alpha - T_z^*) \\ -(s^2 - M_q s) & M_\alpha + T_M^* \end{matrix} = 0$

Page	Line	Correction
53	Equa. IV-11	$\begin{vmatrix} -g & X_u - s & X_\alpha - T_x^* \\ U_0 s & Z_u & Z_\alpha - T_z^* - U_0 s \\ s^2 & -M_u & -(M_\alpha + T_M^*) \end{vmatrix} \begin{Bmatrix} e \\ u \\ \alpha \end{Bmatrix} = \begin{Bmatrix} T_x \\ T_z \\ T_M \end{Bmatrix} \lvert \lambda \rvert$
72	13 down	$X_0 = -1/2\, \rho U^2 S\left[C_{D_{MIN}} + C_{L_\alpha}^2 (\alpha_0 - \alpha_{0_L})^2 / \pi A\right]$
72	Equa. B-4	$T_c \cos\lambda_0 - 1/2\, \rho U^2 S\left[C_{D_{MIN}} + \frac{C_{L_\alpha}^2 (\alpha_0 - \alpha_{0_L})^2}{\pi A}\right] = 0$
72	Equa. B-6, 4th term	$-\left[T_0 \,(e \cos\lambda_0 + a \sin\lambda_0) -\right.$
73	line 5 down, 1st term	$\left.-2(a/e)\alpha_{0_L}\right]\alpha_0 -$
73	line 6 down, 1st term	$+(a/e)\,\alpha_{0_L}^2 -$

Figure 9 $\quad \frac{dz}{dt}$ is perpendicular to x-axis

NOTES:

1) "Upwind wave" and "downwind disturbance" both denote a wave traveling upwind; similarly, "downwind wave" and "upwind disturbance" both denote a wave traveling downwind.

2) Appendix B "T_0" in this section is not as mentioned earlier, but is rather the actual cable tension at the vehicle mount.

SUMMARY

Heavy bodies with lifting surfaces are towed behind airplanes in order to isolate certain types of instrumentation, such as magnetometers, from the airplane. The importance of the stability of these vehicles has recently been magnified due to great increase in the sensitivity of such instrumentation.

The coupled motions of the complete system, which consists of the towed vehicle, the cable, and the stable tow airplane, are very complex. This complexity demands the use of certain simplifications in the analyses which also aid the understanding ot the solutions as well. Different simplifying assumptions were studied in order to determine those applicable to the attainment of useful results.

A fairly complete analysis of cable dynamics is included in the study. This analysis leads to the primary simplification that the cable moves from one equilibrium shape to the next while the towed vehicle moves dynamically. A towed vehicle which is believed to have good characteristics is chosen as a starting point. The longitudinal and lateral directional dynamics are then studied both separately and coupled, and the important parameters affecting the stability of the vehicle are isolated. The results are presented in the form of stability boundaries of the dominant parameters.

NOMENCLATURE

a	velocity of a wave transmitted along a cable in a vacuum = T/μ
C_{D_r}	drag coefficient of rope
C	length of cable whose drag = T when normal to the flow
c.g.	center of gravity
D	drag of body
$\bar{d}$	diameter of rope
e	exponential
F	aero force/length to inclined cable
$\bar{F}$	external force vector
g	gravitational constant $\approx$ 32.2 ft/sec/sec
$\bar{h}$	moment of momentum
I_x	moment of inertia about the x-axis
I_{xz}	product of inertia about y-axis
I_y	moment of inertia about the y-axis
I_z	moment of inertia about the z-axis
j	imaginary vector notation $\sim \sqrt{-1}$
K_E	kinetic energy
k	wave number
L	external moment about the x-axis cable length
ℓ	projection of cable in lateral plane cable length
$\bar{M}$	external applied moment vector
M	external moment about the y-axis
M.A.C.	mean aerodynamic chord
m	mass
N	external moment about the z-axis

n	projection of cable in vertical plane
P	roll velocity about x-axis
P_E	potential energy
p	perturbation roll velocity
Q	pitch velocity about y-axis
q	perturbation pitch velocity
R	drag per unit length of cable yaw velocity about z-axis
r	perturbation yaw velocity
S	distance measured along cable wing area
s	Laplace transform operator
T	cable tension
T_L	cable tension rolling moment contribution
T_M	cable tension pitching moment contribution
T_N	cable tension yawing moment contribution
T_x	cable tension component along the x-axis
T_y	cable tension component along the y-axis
T_z	cable tension component along the z-axis
U	velocity along the x-axis
u	perturbation velocity along the x-axis
V	velocity velocity along y-axis
$\bar{V}$	total velocity of towed body
V_o	freestream velocity
v	perturbation velocity along y-axis
W	weight of body velocity along the z-axis

w	weight/length of cable perturbation velocity along the z-axis
XYZ	space fixed inertial axes
xyz	body fixed axes
α	cable weight to drag ratio parameter $\sim \tan^{-1}$ (W/2R) vehicle angle of attack
β	vehicle sideslip angle
γ	wavelength of traveling wave along cable flight path angle
ϵ	phase angle
η	frequency of traveling wave along cable non-dimensional height of a point on the cable $\sim$ y/c
Λ	angle between cable and relative wind (See Section IV)
λ	angle between the cable and the relative wind perturbation angle between cable and relative wind (See Section IV)
$\bar{\lambda}$	angle between cable and body centerline
μ	mass density of tow cable
ξ	non-dimensional horizontal distance of a point on the cable from the origin $\sim$ x/c
ρ	air density
σ	non-dimensional distance along the cable from the origin $\sim$ s/c
τ	period of traveling wave along cable cable tension ratio T/T_o
φ	angle of inclination of the cable from the local vertical vehicle roll angle
ψ	vehicle yaw angle
ω	general symbol for frequency $\sim$ radians/second

Subscript

o	equilibrium condition
o_L	zero lift

Other Notation

$\ddot{}$	second derivative (total d^2/dt^2 or partial $\partial^2/\partial t^2$ with respect to time
$\dot{}$	first derivative (total d/dt or partial $\partial/\partial t$) with respect to time
$'$	first partial derivative with respect to a space variable
$''$	second partial derivative with respect to a space variable

I INTRODUCTION

The possibility of utilizing airborne towed vehicles as a solution to a number of problems has existed for many years. These vehicles have ranged from simple weights or drogues at the ends of antennas or refueling lines, through more complex machines containing measurement instrumentation that it is desired to isolate from the main aircraft, to large fully-manned gliders. The present study is limited to unpiloted systems and is primarily oriented toward consideration of the problem of the stability characteristics of vehicles utilizing aerodynamic surfaces.

There have been a number of investigations of particular aspects of the towed vehicle stability problem. For the most part, these investigations have been concerned with either very light towed bodies, such as antennas, where the problem is primarily one of cable dynamics; or with heavy, high drag bodies whose inertia prevented the development of stability problems. Seldom, however, has the problem of a towed vehicle with lifting surfaces, which is light enough to have appreciable dynamic response, but heavy enough to have a predominant effect on the cable dynamics, been considered. In addition, the increasing complexity and sensitivity of the instruments carried in the towed vehicles has made the attainment of some insight into the important design parameters of the vehicle imperative. The study reported here is an attempt to correlate the aspects of both cable and vehicle dynamics into a complete system analysis designed to isolate the important parameters.

The total system consists of the vehicle dynamics, the cable dynamics, and, of course, those of the mother airplane. It has been assumed from the outset that the dynamic properties of the mother airplane

are well known and that it is stable. Consequently, the system of primary interest is that of the combination of cable and vehicle dynamics. This system is depicted in block diagram form in Figure 1.

The simplicity of the figure belies the actual complexity of the system. The input γ_A in Figure 1 is the input to the cable from a disturbance of the mother airplane. This, then, incites a cable vibration, m, which incites the towed vehicle. In addition, the towed vehicle is also subjected to extraneous gust loads u_g which may well be the primary excitation. The combined response of the vehicle, V_r, which is a set of a number of response variables, is then coupled with the original cable excitation to form the summed input to the cable dynamics i_s.

The complexity of the system would make a direct attack on a complete system analysis so involved that the possibility of concealing some of the important parameters would exist. Consequently, the problem was approached through a series of studies which started with very simplified approximations and proceeded stepwise to the more complex cases. Not only did this approach insure that the more important parameters were identified and investigated, but it also resulted in an insight as to the amount of simplification that could be applied and still give reasonable results.

This report follows the same line of attack to some degree. Several studies are presented which may appear elementary or even inapplicable. However, it is felt that their inclusion will lead to a better understanding of the system and of the simplifying assumptions utilized in the attainment of the majority of the results.

A configuration which had evolved through years of experimentation was chosen as a point from which the parametric study could emanate.

As might be expected, the basic configuration displayed very good characteristics. The primary results are presented in the form of stability boundaries which developed as the most convenient method of expressing the effect of variations in the important parameters. Other results, such as the effects of tow line variations, are discussed and comparison plots are presented. The agreement of the responses calculated for the basic configuration with those observed in flight indicates that a degree of confidence may be placed in the results presented, at least so far as an indication of the major trends to be expected.

II CABLE DYNAMICS

The dynamics of cables or strings in vacuums are presented with varying degrees of completeness in many mathematical texts and other treatises such as References 1, 2, and 3. Seldom, however, are these discussions carried to the completeness required in a study of the dynamics of a towed vehicle traveling at speeds of such magnitude that the aerodynamic or hydrodynamic loads on the cable are not negligible. This section is an attempt to correlate the analyses of somewhat simplified vibrating string problems with the more complex towed vehicle problem.

The classical vibrating string problem is given consideration first in order to establish the basic concepts utilized in the more complex problems. The shapes of a cable in equilibrium in an airstream are next developed for both a constant tension and a variable tension cable. These shapes are, in effect, the shape of the fundamental mode of a cable in a vacuum with a distributed forcing function equivalent to the aerodynamic force.

These developments are followed by the consideration of a straight cable at an angle to the airstream and the great increase in the complexity of the problem resulting from the addition of the aerodynamic forces. This leads, then, to the formulation of the complete problem and the applicable simplifying assumptions which allow the attainment of useful results.

Dynamics of a Constant Tension, Weightless Cable in a Vacuum

The transverse equation of motion of a weightless cable in a vacuum may be obtained from a consideration of an element of a cable disturbed from rest in a vacuum as shown in Figure 2. The mass density of

the string is μ so that the mass of an element dS of the string is μdS. If the string at rest lies along the x-axis of an x-z orthogonal axis system, the equation expressing the motion of the string in the z-direction is

$$\mu dS\ \partial^2 z/\partial t^2 = (T + dT)\ \sin\ (\lambda + d\lambda) - T\ \sin\ \lambda$$

where T is the tension in the string, and dS is the elemental length. Assuming the line stretch during the vibration to be negligible, dS is approximately a constant, and $dT = 0$. The equation of motion is then

$$\mu dS\ \partial^2 z/\partial t^2 = T\ \cos\ \lambda\ d\lambda$$

Since the line stretch is small, the angle λ must be small and

$$\cos\ \lambda = \frac{1}{\sqrt{1 + \mathrm{Tan}^2\lambda}} \approx 1$$

when the square of the tangent of a small angle is neglected compared to 1. Also,

$$d\lambda = \partial\lambda/\partial S\ dS$$

$$\lambda \approx \sin\ \lambda\quad \frac{\mathrm{Tan}\ \lambda}{\sqrt{1 + \mathrm{Tan}^2\lambda}} \approx \mathrm{Tan}\ \lambda = \frac{\partial z}{\partial x}$$

and since $dS \approx dx$

$$d\lambda = \frac{\partial(\partial z/\partial x)}{\partial x}\ dS = \frac{\partial^2 z}{\partial x^2}\ dS$$

The equation of motion is then

$$\frac{\partial^2 z}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 z}{\partial x^2}$$

or, defining $T/\mu = a^2$

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 z}{\partial t^2} \qquad \text{II-1}$$

Equation II-1 is seen to be of the form of the well known wave equation. The most general solution was first developed by D'Alembert and is known

to be

$$z = f(x - at) + g(x + at) \qquad \text{II-2}$$

This general solution may be modified to apply to any particular case provided the restrictions due to the assumptions used in developing Equation II-1 are not violated.

Infinite String - Traveling Waves

Consider an infinite string which extends from $-\infty$ to $+\infty$ along the x-axis with the conditions that $z = \Phi(x)$ and $\partial z/\partial t = \Psi(x)$ when $t = 0$. Application of the initial conditions to Equation II-2 yields:

$$z(x, 0) = g(x) + f(x) = \Phi(x) \qquad \text{II-3a}$$

$$\left.\frac{\partial z}{\partial t}\right|_{t=0} = a\,[g'(x) - f'(x)] = \Psi(x) \qquad \text{II-3b}$$

Integrating Equation II-3b gives

$$g(x) - f(x) = 1/a \int_0^x \Psi(\xi)\, d\xi + k \qquad \text{II-4}$$

Combining Equations II-3a and II-4 yields

$$g(x) = 1/2\, \Phi(x) + 1/2a \int_0^x \Psi(\xi)\, d\xi + 1/2\, k \qquad \text{II-5a}$$

$$f(x) = 1/2\, \Phi(x) - 1/2a \int_0^x \Psi(\xi)\, d\xi - 1/2\, k \qquad \text{II-5b}$$

Substitution of Equations II-5a and b into Equation II-2 then yields the solution corresponding to the given initial conditions:

$$z = 1/2\, \left[\Phi(x - at) + \Phi(x + at) + 1/a \int_{x-at}^{x+at} \Psi(\xi)\, d\xi\right] \qquad \text{II-6}$$

The most useful wave form in the study of cable dynamics is the harmonic wave. If the wave profile at $t = 0$ is given by

$$\Phi(x) = b \cos mx$$

Then at any time t, the displacement z is

$$z = b \cos m(x - at) \qquad \text{II-7}$$

The following parameters of this wave may, then, be defined

wavelength	$\gamma = 2\pi/m$
period	$\tau = \gamma/a$
frequency	$\eta = 1/\tau$
propagation velocity	$a = \eta\gamma$
wave number	$k = 1/\gamma$

Incorporating these definitions, Equation II-7 may be written as

$$z = b \cos 2\pi(kx - \eta t) \qquad \text{II-8a}$$

or

$$z = A\, e^{2\pi j(\eta t - kx)} \qquad \text{II-8b}$$

where $A = b\, e^{-j\epsilon}$ if the wave is out of phase with some reference by the angle ϵ, or $A = b$ if there is no phase difference.

The kinetic energy of a vibrating string may be determined from the velocity $\partial z/\partial t$ of any point of the string. It is

$$K_E = 1/2 \int \mu(\dot{z})^2 dS = 1/2 \int \mu(\dot{z})^2 dx \qquad \text{II-9}$$

where $\dot{z} = dz/dt$. The potential energy stored in the string may be found by considering the stretch in an element of the string from dx to dS. The amount of work done is then T(dS - dx) and the potential energy is

$$P_E = \int T(dS - dx)$$

or, since $dS = \sqrt{dx^2 + dz^2}$

$$P_E = \int T\left[\sqrt{1 + (\partial z/\partial x)^2} - 1\right] dx$$

Furthermore, since the slope of the string is assumed to always be small, the square root may be approximated by

$$\sqrt{1 + (\partial z/\partial x)^2} = 1 + 1/2\, (\partial z/\partial x)^2$$

and

$$P_E \approx 1/2 \int T(\partial z/\partial x)^2 dx \qquad \text{II-10}$$

With a positive running traveling wave, $z = f(x - at)$, the energies become

$$K_E = 1/2\ \mu\ a^2 \int (f')^2\ dx = 1/2\ T \int (f')^2 dx \qquad \text{II-11a}$$

$$P_E = 1/2\ T \int (f')^2 dx \qquad \text{II-11b}$$

and they are equal.

Finite String - Fixed Ends

When the ends of the cable are fixed, the solution may be obtained by requiring that Equation II-6 satisfy appropriate boundary conditions as long as they are linear. If more complicated situations are anticipated, it is perhaps more informative to consider even the simple cases by utilizing the method of separation of variables.** This method is described for this situation in References 1 and 2 and many other texts and is therefore only included here in outline form.

Considering the basic wave equation, Equation II-1, with initial conditions

$$\begin{aligned} z(x, 0) &= \phi(x) \\ \partial z/\partial t\,\big|_{t=0} &= \Psi(x) \end{aligned} \qquad \text{II-12}$$

and fixed and boundary conditions

$$\begin{aligned} z(0, t) &= 0 \\ z(\ell, t) &= 0 \end{aligned} \qquad \text{II-13}$$

the assumption of a particular solution of the form

$$z(x, t) = F(x)G(t) \qquad \text{II-14}$$

and substitution into Equation II-1 yields:

**Other methods such as the Laplace transform method may also be used. In general, however, complicated initial or boundary conditions greatly increase the complexity of the applications and they are not used here for that reason.

$$\frac{F''(x)}{F(x)} = \frac{1}{a^2}\frac{\ddot{G}(t)}{G(t)} \qquad \text{II-15}$$

The left side of Equation II-15 is now independent of t and the right side is independent of x. Consequently, they must both be equal to a constant, say $-p^2$.** Two equations then obtained.

$$F''(x) = -p^2F(x) \qquad \text{II-16a}$$

$$\ddot{G}(t) = -p^2a^2G(t) \qquad \text{II-16b}$$

Equations II-16a and b are linear differential equations with constant coefficients. Application of Equations II-12 and II-13 to Equation II-14 yields the boundary conditions for Equation II-16a as:

$$F(0) = F(\ell) = 0 \qquad \text{II-17}$$

The solution to Equation II-16a is then

$$F(x) = A\cos px + B\sin px \qquad \text{II-18}$$

Application of the boundary conditions of Equation II-17 then yields:

$$\text{at } x = 0, \quad F(0) = A = 0 \qquad \text{II-19a}$$

$$\text{at } x = \ell, \quad F(\ell) = B\sin p\ell = 0 \qquad \text{II-19b}$$

Equation II-19b is satisfied when $p\ell = n\pi$ where $(n = 0, 1, 2, \ldots)$. Consequently, the solution to Equation II-16a takes the form

$$F(x) = B\sin n\pi/\ell\, x \qquad \text{II-20}$$

The solution to Equation II-16b may then be obtained as

$$G(t) = C\cos pat + D\sin pat$$

or upon substitution of $p = n\pi/\ell$

$$G(t) = C\cos n\pi a/\ell\, t + D\sin n\pi a/\ell\, t \qquad \text{II-21}$$

Substitution of Equations II-20 and II-21 into Equation II-14 yields solutions in two forms.

**In general, both $+p^2$ and 0 could also yield solutions. However, in this case, it can be shown that these values of the constant do not satisfy the boundary conditions.

$$z(x, t) = E \sin n\pi x/\ell \cos n\pi a/\ell \, t$$

or

$$z(x, t) = D \sin n\pi x/\ell \sin n\pi a/\ell \, t$$

Either of these forms may be taken as the solution or any linear combination may be taken as the solution. Therefore, the solutions may be written:

$$z(x, t) = \sum_{n=1}^{\infty} E_n \sin n\pi x/\ell \cos n\pi a/\ell \, t \qquad \text{II-22a}$$

or

$$z(x, t) = \sum_{n=1}^{\infty} D_n \sin n\pi x/\ell \sin n\pi a/\ell \, t \qquad \text{II-22b}$$

From equation II-22a

$$\frac{\partial z}{\partial t} = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi x}{\ell} \left[- \frac{k\pi a}{\ell} \sin \frac{k\pi a}{\ell} \, t \right]$$

and at $t = 0$

$$\partial z/\partial t \big|_{t=0} = 0$$

Therefore, it is seen from Equation II-12 that solution II-22a would satisly the initial condition

$$\partial z/\partial t \big|_{t=0} = \Psi(x) = 0 \qquad \text{II-23}$$

Evaluation of $z(x, 0)$ from Equation II-22a yields

$$z(x, 0) = \sum_{n=1}^{\infty} E_n \sin n\pi x/\ell = \Phi(x) \qquad \text{II-24}$$

A closed form solution for E_n cannot be obtained from Equation II-24. However, $\Phi(x)$ is an odd function of period 2ℓ and may be expanded in a Fourier series of the form

$$\Phi(x) = \sum_{r=1}^{\infty} b_r \, r\pi x/\ell$$

A comparison with Equation II-24 indicates $b_r = E_n$ when $r = n$. E_n is therefore obtained by substituting n for r in the half range formula for the coefficient of an odd function.

$$E_n = 2/\ell \int_0^\ell \Phi(x) \sin n\pi x/\ell \, dx \qquad \text{II-25}$$

Equation II-22a is then the solution of the problem with initial conditions

$$z(x, 0) = \Phi(x)$$

$$\partial z/\partial t\big|_{t=0} = 0$$

and E_n as given in Equation II-25.

A consideration of Equation II-22b reveals that

$$z(x, 0) = 0$$

and

$$\left.\frac{\partial z}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} D_n \frac{n\pi a}{\ell} \sin \frac{n\pi x}{\ell} = \Psi(x) \qquad \text{II-26}$$

A similar Fourier analysis of Equation II-26 then yields

$$\frac{n\pi a}{\ell} D_n = \frac{2}{\ell} \int_0^\ell \Psi(x) \sin \frac{n\pi x}{\ell} \, dx \qquad \text{II-27}$$

Equation II-22b with D_n evaluated from Equation II-27 is then the solution of the problem with initial conditions

$$z(x, 0) = 0$$

$$\partial z/\partial t\big|_{t=0} = \Psi(x)$$

A solution of the problem with the initial conditions of Equation II-12 is then a linear combination of the solutions given as Equations II-22a and II-22b with the coefficients evaluated from Equations II-25 and II-27.

Forced Vibration

When the string is subjected to an external force, the equation of motion is non-homogeneous and has the form

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial t^2} - \frac{\partial^2 z}{\partial x^2} = F(x, t) \qquad \text{II-28}$$

where $F(x, t)$ is the external force per unit length divided by the tension in the string. The boundary conditions correspond to fixed ends

$$z(0, t) = z(\ell, t) = 0 \qquad \text{II-29}$$

and the initial conditions are here specialized to correspond to a string at rest

$$z(x, 0) = \Phi(x)$$
$$\partial z/\partial t \big|_{t=0} = 0 \qquad \text{II-30}$$

for simplicity. The same procedure would, however, handle the general initial conditions of Equation II-12.

The problem is no longer homogeneous and a solution cannot be derived from the sum of several solutions. However, a solution can be composed of a transient part plus a particular integral. The total solution of Equation II-28, with boundary and initial conditions as given in Equations II-29 and II-30, has the form

$$z(x, t) = z_0(x, t) + z_1(x, t) \qquad \text{II-31}$$

$z_0(x, t)$ is the solution of Equation II-1 with the boundary conditions

$$z(0, t) = z(\ell, t) = 0$$

and initial conditions

$$z(x, 0) = \Phi(x)$$
$$\partial z/\partial t \big|_{t=0} = 0$$

$z_1(x, t)$ is the solution of Equation II-28 with the boundary conditions

$$z(0, t) = z(\ell, t) = 0$$

and the initial conditions

$$z(x, 0) = \partial z/\partial t \big|_{t=0} = 0$$

The solution $z_0(x, t)$ was presented previously as Equation II-22a; only $z_1(x, t)$, then, is left to be obtained.

The forcing function $F(x, t)$ may be expanded as a Fourier series at any given time.

$$F(x, t) = \sum_{k=1}^{\infty} A_k(t) \sin k(\omega_0/a)x \qquad \text{II-32}$$

Similarly, it may be assumed that

$$z_1(x, t) = \sum_{k=1}^{\infty} B_k(t) \sin k(\omega_c/a)x \qquad \text{II-33}$$

Substitution of Equations II-33 and II-32 into Equation II-28 after dropping the summation signs for ease of notation, yields:

$$\ddot{B}_k(t) + B_k(t)k^2\omega_0{}^2 = a^2 A_k(t) \qquad \text{II-34}$$

with the initial conditions

$$B_k(0) = \dot{B}_k(0) = 0 \qquad \text{II-35}$$

The solution of the homogeneous portion of Equation II-34 is then

$$B_k(t) = C_{k_1} \cos k\omega_0 t + C_{k_2} \sin k\omega_0 t \qquad \text{II-36}$$

The method of variation of parameters then yields the particular solution to Equation II-34 which when combined with Equation II-36 gives the complete solution as

$$B_k(t) = D_{k_1}(t) \cos k\omega_0 t + D_{k_2}(t) \sin k\omega_0 t \qquad \text{II-37}$$

Then, at $t = 0$, from Equation II-35

$$\begin{aligned} B_k(0) &= 0 = D_{k_1}(0) \\ \dot{B}_k(0) &= 0 = D_{k_1}(0) + k\omega_0 D_{k_2}(0) \end{aligned} \qquad \text{II-38}$$

Evaluation of the first and second time derivatives of Equation II-37, substitution into Equation II-34 and multiplication of both sides of the resulting equation by $\cos k\omega_0 t$, yields:

$$\begin{aligned} &\ddot{D}_{k_1}(t) \cos^2 k\omega_0 t - 2k\omega_0 \dot{D}_{k_1}(t) \sin k\omega_0 t \cos k\omega_0 t + \ddot{D}_{k_2}(t) \cos k\omega_0 t \\ &\sin k\omega_0 t + 2k\omega_0 \dot{D}_{k_2}(t)\cos^2 k\omega_0 t = a^2 A_k(t) \cos k\omega_0 t \end{aligned} \qquad \text{II-39}$$

Substitution of $d/dt\,[\dot{D}_{k_1}(t)\cos^2 k\omega_0 t]$ for the first two terms in Equation II-39 and $d/dt\,[\dot{D}_{k_2}(t) \sin k\omega_0 t \cos k\omega_0 t] + k\omega_0 \dot{D}_{k_2}(t)$ for the last two terms; this becomes

$$d/dt\,[\dot{D}_{k_1}(t)\cos^2 k\omega_0 t] + d/dt\,[\dot{D}_{k_2}(t)\sin k\omega_0 t\cos k\omega_0 t] + k\omega_0\,\dot{D}_{k_2}(t) = a^2 A_k(t)\cos k\omega_0 t$$

Integration yields

$$\dot{D}_{k_1}(t)\cos^2 k\omega_0 t + \dot{D}_{k_2}(t)\sin k\omega_0 t\cos k\omega_0 t + k\omega_0 D_{k_2}(t) = a^2\int_0^t A_k(\tau)\cos k\omega_0\,\tau d\tau$$

where the constant of integration is zero. Division by $\cos^2 k\omega_0 t$ and substitution of $d/dt\,[D_{k_2}(t)\,\frac{\sin k\omega_0 t}{\cos k\omega_0 t}]$ for the resulting second and third terms then gives

$$\dot{D}_{k_1}(t) + \frac{d}{dt}\left[D_{k_2}(t)\,\frac{\sin k\omega_0 t}{\cos k\omega_0 t}\right] = \frac{a^2}{\cos^2 k\omega_0 t}\int_0^t A_k(\tau)\cos k\omega_0\tau d\tau$$

Another integration results in

$$D_{k_1}(t) + D_{k_2}(t)\,\frac{\sin k\omega_0 t}{\cos k\omega_0 t} = a^2\int_0^t \frac{d\tau'}{\cos^2 k\omega_0\tau'}\int_0^{\tau'} A_k(\tau)\cos k\omega_0\tau d\tau$$

Evaluation of the right hand side by integrating by parts gives

$$D_{k_1}(t) + D_{k_2}(t)\,\frac{\sin k\omega_0 t}{\cos k\omega_0 t} = \frac{a^2}{\cos k\omega_0 t}\int_0^t A_k(\tau)\sin k\omega_0(t-\tau)d\tau \qquad \text{II-40}$$

Multiplication of Equation II-40 by $\cos k\omega_0 t$ and substitution into Equation II-37 gives the desired complete solution of Equation II-34 as:

$$B_k(t) = a^2\int_0^t A_k(\tau)\sin k\omega_0(t-\tau)d\tau \qquad \text{II-41}$$

Substitution of Equation II-41 into II-33 after changing the period from 2π to 2ℓ yields the steady state solution of Equation II-28.

$$z_1(x,\,t) = \sum_{k=1}^{\infty}\left[a^2\int_0^t A_k(\tau)\sin ak\pi/\ell\,(t-\tau)d\tau\right]\sin k\,\pi/\ell\,x \qquad \text{II-42}$$

From Equation II-32 with the substitution $\omega_0 = a\pi/\ell$, it is seen that $A_k(t)$ is given by

$$A_k(t) = 2/\ell\int_0^{\ell} F(x,\,t)\sin k\pi/\ell\,x\,dx \qquad \text{II-43}$$

This result holds for any force $F(x, t)$ as long as the force can be expanded in a Fourier series. An example would be a sinusoidal forcing function $F_1 \sin \omega t$ applied at time $t = 0$ at some point x_0 of a string at rest along the x-axis. $\Phi(x) = \Psi(x) = 0$ for this condition, and the complete solution is given by Equation II-42. $F(x, t)$ of Equation II-43 was taken as a distributed force. Therefore, $F_1 \sin \omega t$ must be considered to act over a small element Δx of the string between $x = x_0$ and $x = x_0 + \Delta x$ such that

$$F_1 \sin \omega t = F_0 \Delta x \sin \omega t$$

then

$$F(x, t) = F_1/\Delta x \sin \omega t$$

and

$$A_k(t) = 2F_1/\Delta x \ell \int_{x_0}^{x_0+\Delta x} \sin \omega t \sin k\pi/\ell \, x \, dx \qquad \text{II-44}$$

Integration yields

$$A_k(t) = \frac{-2F_1}{\Delta x k\pi} \sin \omega t[\cos \frac{k\pi}{\ell} x_0(\cos k\pi/\ell \, \Delta x - 1) - \sin k\pi/\ell x_0 \sin k\pi/\ell \, \Delta x]$$

Then the limit as $\Delta x \to 0$ gives

$$A_k(t) = 2F_1/\ell \sin k\pi/\ell \, x_0 \sin \omega t \qquad \text{II-45}$$

Substitutions into Equation II-42 then gives $z(x, t)$.

$$z(x, t) = \frac{2F_1 a^2}{\ell} \sum_{k=1}^{\infty} \sin k\pi/\ell \, x_0 \sin k\pi/\ell \, x \int_0^t \sin \omega t \sin ak\pi/\ell \, (t-\tau) \, d\tau$$

Evaluation of the integral then yields the final result

$$z(x, t) = \frac{2F_1 a^2}{\ell} \sum_{k=1}^{\infty} \sin \frac{k\pi}{\ell} x_0 \sin \frac{k\pi}{\ell} x \left[\frac{\omega \sin ak\pi/\ell \, t - ak\pi/\ell \sin \omega t}{\omega^2 - (ak\pi/\ell)^2}\right] \qquad \text{II-46}$$

Time Dependent Boundary Conditions

Time dependent boundary conditions occur whenever the ends of the string are not rigidly fixed, but are restricted in some manner which either varies with time directly or is dependent upon an input from the

string. The situation which exists at a variable boundary lies between that of an infinite string problem and a fixed boundary problem, for some, but not all, of the energy is reflected while the remainder is absorbed by the boundary.

The method of separation of variables used previously no longer applies when the boundary conditions are time dependent, for it is impossible to satisfy the boundary conditions by adjusting only space dependent functions. However, by applying a procedure developed by R. D. Mindlin and L. E. Goodman in Reference 4, it is possible to separate the solution into two parts, one of which is later adjusted to satisfy the boundary conditions on the other.

For a string initially at rest along the x-axis, the equation of motion in the absence of an externally applied force is given in Equation II-1 as

$$\frac{\partial^2 z}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 z}{\partial t^2} = 0 \qquad \text{II-1}$$

and the initial conditions are

$$z(x,\ 0) = \dot{z}(x,\ 0) = 0$$

The boundary conditions considered vary directly with time at each end and may be expressed as

$$\begin{aligned} z(0,\ t) &= Ae^{-j\omega_0 t} = \beta_1(t) \\ z(\ell,\ t) &= Be^{-j\omega_\ell t} = \beta_2(t) \end{aligned} \qquad \text{II-47}$$

A solution of Equation II-1 may then be chosen as

$$z(x,\ t) = \zeta(x,\ t) + \sum_{i=1}^{\infty} \beta_i(t)\ \alpha_i(x) \qquad \text{II-49}$$

From a substitution of Equation II-49 into Equation II-1, it is seen that $\zeta(x,\ t)$ must satisfy the expression

$$a^2 \frac{\partial^2 \zeta}{\partial x^2} - \frac{\partial^2 \zeta}{\partial t^2} = - \sum_{i=1}^{2} \left[a^2 \beta_i(t)\, \alpha_i''(x) - \ddot{\beta}_i(t)\, \alpha_i(x) \right] \qquad \text{II-50}$$

with the boundary conditions

$$\begin{aligned} \zeta(0, t) &= \beta_1(t) - \sum_{i=1}^{2} \beta_i(t)\, \alpha_i(0) \\ \zeta(\ell, t) &= \beta_2(t) - \sum_{i=1}^{2} \beta_i(t)\, \alpha_i(\ell) \end{aligned} \qquad \text{II-51}$$

and the initial conditions

$$\begin{aligned} \zeta(x, 0) &= - \sum_{i=1}^{2} \beta_i(0)\, \alpha_i(x) \\ \dot{\zeta}(x, 0) &= - \sum_{i=1}^{2} \dot{\beta}_i(0)\, \alpha_i(x) \end{aligned} \qquad \text{II-52}$$

Consequently, it may be seen that if the $\alpha_i(x)$ are chosen such that the right hand side of Equations II-51 are zero, Equation II-50 may be solved by the method of separation of variables. This requires that the relations

$$\begin{aligned} \alpha_1(0) &= 1 \qquad & \alpha_2(0) &= 0 \\ \alpha_1(\ell) &= 0 \qquad & \alpha_2(\ell) &= 1 \end{aligned} \qquad \text{II-53}$$

be satisfied. The $\alpha_i(x)$ may then be chosen as polynomials in x or summations of trigonometric or exponential terms whose coefficients are determined from the relations II-53.

With the boundary conditions of Equation II-48 certain simplifications are attained if the $\alpha_i(x)$ are taken as summations of trigonometric terms.

$$\begin{aligned} \alpha_1(x) &= a_1 \sin \omega_0/a\, x + b_1 \cos \omega_0/a\, x \\ \alpha_2(x) &= a_3 \sin \omega_\ell/a\, x + b_3 \cos \omega_\ell/a\, x \end{aligned} \qquad \text{II-54}$$

Substitution of Equations II-54 into Equations II-53 yields the $\alpha_i(x)$ as:

$$\begin{aligned} \alpha_1(x) &= \cos \omega_0/a\, x - \cot \omega_0/a\, \ell \sin \omega_0/a\, x \\ \alpha_2(x) &= \csc \omega_\ell/a\, \ell \sin \omega_\ell/a\, x \end{aligned} \qquad \text{II-55}$$

Substitutions of Equations II-55 into Equations II-51, II-52 and II-50 give the desired boundary conditions

$$\zeta(0, t) = \zeta(\ell, t) = 0 \qquad \text{II-56}$$

The initial conditions

$$\zeta(x, 0) = -A\cos(\omega_0/a\ x) + A\cot(\omega_0/a\ \ell)\sin(\omega_0/a\ x) - B\csc(\omega_\ell/a\ \ell)\sin(\omega_\ell/a\ x) \qquad \text{II-57a}$$

$$\dot{\zeta}(x, 0) = Aj\omega_0[\cos(\omega_0/a\ x) - \cot(\omega_0/a\ \ell)\sin(\omega_0/a\ x)] + Bj\omega_\ell\csc(\omega_\ell/a\ \ell)\sin(\omega_\ell/a\ x) \qquad \text{II-57b}$$

and the differential equation

$$a^2\frac{\partial^2\zeta}{\partial x^2} - \frac{\partial^2\zeta}{\partial t^2} = 0 \qquad \text{II-58}$$

The solution of Equation II-58 with boundary conditions of Equation II-56 and initial conditions of Equations II-57 may now be obtained by separation of variables and is the combination of the previously presented Equations II-22a and II-22b.

$$\zeta(x, t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{\ell}\left[E_n\cos\frac{n\pi a}{\ell}t + D_n\sin\frac{n\pi a}{\ell}t\right] \qquad \text{II-59}$$

Substitution of Equation II-57a for $\Phi(x)$ and Equation II-57b for $\Psi(x)$ in Equations II-25 and II-27, and performance of the necessary integrations yields the coefficients

$$E_n = \frac{(-1)^n\, 2Bn\pi}{\ell^2[(n\pi/\ell)^2 - (\omega_\ell/a)^2]} \quad ; \quad D_n = -\frac{(-1)^n\, 2Bj(\omega_\ell/a)}{\ell[(n\pi/\ell)^2 - (\omega_\ell/a)^2]} \qquad \text{II-60}$$

The solution, obtained from substitutions of Equations II-60 into Equation II-59 and the result along with the $\alpha_i(x)$ and $\beta_i(t)$ into Equation II-49, is then

$$z(x, t) = A[\cos(\omega_0/a\ x) - \cot(\omega_0/a\ \ell)\sin(\omega_0/a\ x)]e^{-j\omega_0 t} + Be^{-j\omega_\ell t}\cdot\csc(\omega_\ell/a\ \ell)\sin(\omega_\ell/a\ x) + \sum_{n=1}^{\infty}\frac{2(-1)^n B}{\ell[(n\pi/\ell)^2 - (\omega_\ell/a)^2]}[n\pi a/\ell\cos(n\pi a/\ell\ t) - j\,\omega_\ell/a\sin(n\pi a/\ell\ t)]\sin(n\pi/\ell\ x) \qquad \text{II-61}$$

The denominator of the summation in Equation II-61 is seen to vanish when the end frequency ω_ℓ is

$$\omega_\ell = an\pi/\ell$$

If the corresponding mode (n) is excited, $z(x, t) \to \infty$ and the frequency ω_ℓ is known as the n^{th} modal resonance frequency.

<u>The Development of the Static Equilibrium Equations of a Weightless Cable Supporting a Body in a Constant Velocity Flow Field</u>

The development here of the equilibrium equations of a wieght-less cable supporting a body in a flow field follows the procedure described in Reference 5.

The wire is taken to be weightless and inextensible. R is the drag per unit length of the cable when at right angles to the flow.

$$R = C_{D_R} \bar{d} \rho V^2 \qquad \text{II-62}$$

When the cable is inclined at an angle, λ, (see Figure 3) to the free stream, the force per unit length of the cable at right angles to the cable is of the following form;

$$F = R \sin^2 \lambda \qquad \text{II-63}$$

Since the aerodynamic force on the cable is perpendicular to each element of length, and because it is weightless, the tension in the cable will be constant, denoted by T, and has the following form;

$$T \partial\varphi/\partial S = F \qquad \text{II-64}$$

Where ∂S is an element of length, and φ is the inclination of ∂S from the vertical (see Figure 3). The tension may also be expressed

$$T = Rc \qquad \text{II-65}$$

where c is the length of wire whose drag, when normal to the flow, is equal to the tension T. The differential equation for the shape of the wire becomes

$$dS/d\varphi = c \sec^2 \varphi \qquad \text{II-66}$$

Next, the wire may be considered to be supporting a body of weight T and zero drag. The system is illustrated in Figure 4, where the origin O is taken at the bottom point of the wire. Integrating Equation II-66

$$S = c \tan \varphi \qquad \text{II-67}$$

The rectangular coordinates of any point of the wire with reference to the origin may be derived from the differential equations,

$$dx/dS = \sin \varphi \qquad \text{II-68}$$

$$dy/dS = \cos \varphi \qquad \text{II-69}$$

Integrating these expressions with respect to the origin at O as before,

$$S = c \tan \varphi = \sinh y/c, \qquad \text{II-70}$$

$$x = c(\cosh \eta - 1). \qquad \text{II-71}$$

Defining

$$\sigma = S/c, \; \xi = x/c, \; \eta = y/c$$

Equations II-70 and II-71 can be rewritten as follows

$$\sigma = \tan \varphi = \sinh \eta, \qquad \text{II-72}$$

$$\xi = \cosh \eta - 1. \qquad \text{II-73}$$

Next, the length of wire BA as shown in Figire 5 may be considered. At B the cable tension T has horizontal and vertical components capable of supporting a body of weight W and drag D provided,

$$W = Rc \cos \varphi \qquad \text{II-74}$$

$$D = Rc \sin \varphi \qquad \text{II-75}$$

The shape of the cable supporting a body may be determined by using Equations II-72, 73, 74, and 75 in the following manner. The values at point A are denoted by subscript 1, and B by subscript 2.

$$\sigma_1 = \sinh \eta_1 = \tan \varphi = D/W \qquad \text{II-76}$$

$$\sigma_2 = \sinh \eta_2 \qquad \text{II-77}$$

$$\xi_1 = \cosh \eta_1 - 1 = \sec \varphi - 1 \qquad \text{II-78}$$

$$\xi_2 = \cosh \eta_2 - 1 \qquad \text{II-79}$$

and

$$\xi_2 - \xi_1 = \alpha = a/c \qquad \text{II-80}$$

$$\eta_2 - \eta_1 = \beta = b/c \qquad \text{II-81}$$

$$\sigma_2 - \sigma_1 = \sigma = S/c \qquad \text{II-82}$$

A combination of these equations will then eliminate the coordinates $(\xi, \eta, \sigma)_1$ and $(\xi, \eta, \sigma)_2$ as

$$\sigma + \tan \varphi = \sinh (\beta + \eta_1) = \sinh (\beta \sec \varphi) + \cosh (\beta \tan \varphi) \qquad \text{II-83}$$

and

$$\alpha + \sec \varphi = 1 + \xi_2 = \cosh (\beta + \eta_1) = \cosh (\beta \sec \varphi) + \sinh (\beta \tan \varphi) \qquad \text{II-84}$$

Thus the two equations which determine the shape of the weightless cable are:

$$\sigma = \sinh (\beta \sec \varphi) + [\cosh (\beta - 1)] \tan \varphi \qquad \text{II-85}$$

$$\alpha = \sinh (\beta \tan \varphi) + [\cosh (\beta - 1)] \sec \varphi \qquad \text{II-86}$$

<u>General Analysis of the Equations Governing the Shape of a Heavy Cable Towing a Heavy Body</u>

The development of the equations governing the equilibrium shape of a heavy cable in an airstream follows the procedures of References 6 and 7. As to be expected, this development parallels that of the weightless cable in many respects. However, certain basic differences do exist; and since the resulting expressions describe the actual physical situation, they are developed here.

The cable is considered to have a weight w per unit length and a drag R per unit length when perpendicular to the free stream velocity. R has the form,

$$R = C_{D_R} \bar{d} \rho V^2 \qquad \text{II-62}$$

where C_{D_R} is the drag coefficient of the cable and d is the diameter. When inclined to the free stream at an angle λ (see Figure 6). The aerodynamic force per unit length, F, perpendicular to the cable, will be of magnitude,

$$F = R \sin^2 \lambda \qquad \text{II-87}$$

For convenience the development will be made first for a heavy body with zero drag and then for a heavy body with drag.

Taking 0 as the origin of coordinates (see Figure 6) with λ as defined earlier, the following relations may be established,

$$\partial x/\partial S = \cos \lambda \qquad \text{II-88}$$

$$\partial z/\partial S = \sin \lambda \qquad \text{II-89}$$

where ∂S is the element of length inclined at an angle λ. The element ∂S, at point P, is in equilibrium and thus the forces along the element of length may be written

$$\partial T/\partial S = w \sin \lambda \qquad \text{II-90}$$

which, when integrating S from the origin to point P, gives

$$T = T_0 + wy \qquad \text{II-91}$$

where T_0 is the weight of the body being supported.

Resolving forces perpendicular to the element of length, the second fundamental equation is obtained

$$T \partial\varphi/\partial S = w \cos \lambda - R \sin^2 \lambda \qquad \text{II-92}$$

At this point it is convenient to introduce the non-dimensionalizing coefficients defined as follows

$$\tau \equiv T/T_o, \quad \eta \equiv \frac{z}{T_o/R}, \quad \xi \equiv \frac{x}{T_o/R}, \quad T \equiv \frac{s}{T_o/R} \qquad \text{II-93}$$

where T_o/R is the length of cable whose drag when perpendicular to the free stream is equal to the tension T_o. Finally, the weight to drag ratio os the cable μ is defined as:

$$w/R = \mu = 2 \tan \alpha \qquad \text{II-94}$$

Utilizing the above notation, Equations II-90 and II-92 become

$$\tau = 1 + \mu\eta \qquad \text{II-95}$$

and

$$\tau\, \partial\lambda/\partial\sigma = \mu \cos \lambda - \sin^2 \lambda \qquad \text{II-96}$$

Elimination of τ from these two equations yields the relationship between η and λ. The corresponding values of ξ and σ are found by evaluating the integrals

$$\xi = \int_0^\eta \cot \lambda\, \partial\eta \qquad \text{II-97}$$

and

$$\sigma = \int_0^\eta \operatorname{cosec} \lambda\, \partial\eta \qquad \text{II-98}$$

For a heavy cable the process is as follows; from Equations II-95 and II-96,

$$(1 + \mu\eta) \sin \lambda\, \partial\ /\partial\eta = \cos^2 \lambda + \mu \cos \lambda - 1 \qquad \text{II-99}$$

which in integral form becomes

$$\int_0^\eta \frac{\mu\partial\eta}{1+\mu\eta} = \int_0^{\pi/2} \frac{\mu \sin \lambda\, \partial\lambda}{1 - \mu\cos\ \ - \cos^2} \qquad \text{II-100}$$

From a substitution of 2 tan α for μ and evaluation of the integrals, it is seen that

$$\log (1 + 2\eta \tan \alpha) = \sin \alpha \log \frac{\cos \alpha + (1 - \sin \alpha) \cos \lambda}{\cos \alpha - (1 + \sin \alpha) \cos \lambda} \qquad \text{II-101}$$

This equation was then used to calculate η as a function of λ. The corresponding values of ξ and σ were obtained by graphical integration of Equations II-97 and II-98.

Next, the equations will be developed for a body with drag. The towed body is assumed to have a net weight of W and a drag D. Net weight is simply the weight minus any lift produced by the body. The angle that the cable makes with the body is determined from the equation

$$\operatorname{Tan} \lambda = W/D \qquad \text{II-102}$$

The tension at this point will be

$$T = \sqrt{W^2 + D^2} \qquad \text{II-103}$$

Knowing σ as a function of λ, the point of connection can now be found; but T_0, the fundamental tension, must still be determined. From Equation II-95, the value of T/T_0 may be determined at any point; and when used in conjunction with Equation II-101, a graph may be constructed to determine T/T_0 as a function of λ directly.

The results of this analysis are presented in graphical form as follows; Figure 7 is simply the cable shape in non-dimensional form, the parameter α being constant for each curve; lines of constant σ are also shown. Figure 8 is a plot of λ versus T_0/T, mentioned earlier.

Dynamics of a Constant Tension, Weightless Line in Moving Air

The aerodynamic force per unit length normal to an element of cable at an angle λ with the airstream is obtained from Equations II-63 and II-62 as:

$$dFa = C_{D_R} \, \bar{d} \, \rho \, V^2 \sin^2 \lambda \, dx \qquad \text{II-104}$$

Consideration of a straight cable initially situated at an angle λ_0 to the airstream velocity V_0 such that the disturbed angle of attack is never negative, shown in Figure 9 where the x-axis is taken along the undisturbed cable, leads to the following relations for V and λ.

$$V^2 = (V_0 - \partial z/\partial t \sin \lambda_0)^2 + (\partial z/\partial t)^2 \cos^2 \lambda_0$$

and since λ_0 is fairly large, $\cos^2 \lambda_0 < 1$ and $(\partial z/\partial t)^2 \cos^2 \lambda_0$ may be neglected when compared with $(V_0 - \partial z/\partial t \sin \lambda_0)^2$. Therefore

$$V \approx V_0 - \partial z/\partial t \sin \lambda_0 \qquad \text{II-105}$$

From Figure 9 it is seen that

$$\lambda = \lambda_0 - \lambda_2 - \lambda_1$$

and

$$\lambda_1 = \sin^{-1} \frac{\partial z/\partial t \cos \lambda_0}{V}$$

and again since $\partial z/\partial t << V$

$$\lambda_1 \approx \frac{\partial z/\partial t \cos \lambda_0}{V}$$

Also,

$$\lambda_2 = \tan^{-1} \partial z/\partial x \approx \sin^{-1} \partial z/\partial x$$

$$\lambda_2 \approx \partial z/\partial x$$

Then

$$\lambda = \lambda_0 - \partial z/\partial x - \partial z/\partial t \cos \lambda_0/V \qquad \text{II-106}$$

or, defining

$$\bar{\lambda} = + \partial z/\partial x + \partial z/\partial t \cos\lambda_0/V$$

$$\lambda = \lambda_0 - \bar{\lambda}$$

Application of the trigonometric identity

$$\sin^2 \lambda = 1/2\ (1 - \cos 2\lambda) = 1/2\ (1 - \cos 2\lambda_0 \cos 2\bar{\lambda} - \sin 2\lambda_0 \sin 2\bar{\lambda})$$

then yields, since $\bar{\lambda} \ll \lambda_o$,

$$\sin^2\lambda = 1/2\ (1 - \cos 2\lambda_o) - \bar{\lambda} \sin 2\lambda_o$$

or

$$\sin^2\lambda = \sin^2\lambda_o - \bar{\lambda} \sin 2\lambda_o$$

or upon substitution for $\dot{\bar{\lambda}}$

$$\sin^2\lambda = \sin^2\lambda_o - (\partial z/\partial x + \partial z/\partial t\ \cos\lambda_o/V) \sin 2\lambda_o \qquad \text{II-107}$$

Substitution of Equations II-107 and II-105 into Equation II-104, upon neglecting higher order small terms, gives the aerodynamic force as

$$dF_a = C_D\ \rho\ \bar{d}\ V_o^2\ [\sin^2\lambda_o - \sin 2\lambda_o\ \partial z/\partial x - 1/V_o\ (\cos\lambda_o \sin 2\lambda_o + 2\sin^3\lambda_o)\ \partial z/\partial t]dx$$

Multiplication of the right hand side of this equation by $\sin 2\lambda_o/\sin 2\lambda_o$ and introduction of the constant parameters

$$\xi = \frac{C_D\ \rho\ \bar{d}\ V_o^2 \sin 2\lambda_o}{T}$$

$$\delta = \cos\lambda_o + \frac{2\sin^3\lambda_o}{\sin 2\lambda_o}$$

then gives the aerodynamic contribution in the form

$$dF_a/T = (\xi \sin^2\lambda_o/\sin 2\lambda_o - \xi\ \partial z/\partial x - 1/V_o\ \xi\ \delta\ \partial z/\partial t)\ dx \qquad \text{II-108}$$

Substitution of the aerodynamic force into Equation II-1 yields the equation of motion.

$$\frac{\partial^2 z}{\partial x^2} + \xi \frac{\sin^2\lambda_o}{\sin 2\lambda_o} - \xi \frac{\partial z}{\partial x} - \frac{1}{V_o} \xi\ \delta \frac{\partial z}{\partial t} = \frac{1}{a^2} \frac{\partial^2 z}{\partial t^2}$$

However, since the string under consideration is initially straight along the x-axis and at rest, $\partial^2 t/\partial x^2 = \partial z/\partial x = \partial^2 z/\partial t^2 = \partial z/\partial t = 0$ initially and it is seen that the constant

$$\xi \sin^2\lambda_o/\sin 2\lambda_o = 0^{**}$$

**For a heavy cable, this expression is $\xi \frac{\sin^2\lambda_o}{\sin 2\lambda_o} = \frac{g}{a^2} \cos\lambda_o$

The equation of motion is then

$$\partial^2 z/\partial x^2 - \xi\, \partial z/\partial x - \xi\delta/V_0\, \partial z/\partial t = 1/a^2\, \partial^2 z/\partial t^2 \qquad \text{II-109}$$

Forced Infinite Cable

The steady state transverse response of a cable forced by a sinusoidal forcing function $Ae^{-j\omega t}$ at a point $x = 0$ will assume the frequency of the forcing function. For a cable which extends to infinity both upwind and downwind of the $x = 0$ point, the resulting displacement along the cable may be assumed to be given by Equation II-8b with η replaced by $\omega/2\pi$.

$$z = Ae^{2\pi j(\omega/2\pi\, t - kx)} \qquad \text{II-110}$$

Substitution of this form into the equation of motion, Equation II-109, yields the desired expression for k.

$$k^2 - j\, \xi/2\pi\, k + 1/4\pi^2(\xi\delta\omega/V_0\, j - \omega^2/a^2) = 0 \qquad \text{II-111}$$

Then, since the resulting expression for k is complex, it may be assumed that

$$k = u + jv$$

Substitution into Equation II-111 yields

$$u^2 + \xi/2\pi\, v - v^2 - \omega^2/4\pi^2 a^2 + j(2uv - \xi/2\pi\, u + \xi\delta\omega/4\pi^2 V_0) = 0 \qquad \text{II-112}$$

Then from the imaginary part of Equation II-112

$$2\pi\, v = \xi/2\, (1 - \delta\omega/2\pi V_0 u) \qquad \text{II-113}$$

Substitution of Equation II-113 into the real part of Equation II-112 then gives

$$2\pi u = \pm [1/2((\omega^2/a^2 - \xi^2/4) \pm [(\omega^2/a^2 - \xi^2/4)^2 - \delta^2\omega^2\xi^2/V_0{}^2]^{1/2})]^{1/2} \qquad \text{II-114}$$

Then from Equation II-110

$$z = Ae^{2\pi vx}\, e^{j(\omega t - 2\pi ux)} \qquad \text{II-115}$$

A downwind disturbance is thus damped when v is negative and amplified when v is positive. The reverse is true for an upwind wave which corresponds to a negative value of u. For a negative value of u, v is always positive and an upwind wave is therefore always damped. The damping of a downwind wave is dependent on the inequality

$$\frac{\delta\omega}{2\pi V_0 u} > 1$$

for damping. Or, since the propagation velocity for this wave is

$$\bar{a} = \frac{\omega}{2\pi u} \qquad \text{II-116}$$

the damping inequality may be expressed

$$\delta \frac{\bar{a}}{V_0} > 1 \qquad \text{II-117}$$

The term δ is in the range $1 \leq \delta \leq \infty$ when λ_0 is in its normal operating range $0 \leq \lambda_0 \leq 90^\circ$. Therefore, for very large λ_0 near 90°, the downwind wave will be damped for all reasonable velocities. However, for λ_0 near zero, the damping depends on $\bar{a}/V_0$. Then, from Equation II-114, as $\lambda_0 \to 0$

$$2\pi u \to \pm \sqrt{1/2\ \omega/a\ (\omega/a \pm 1)}$$

and from Equation II-116

$$\bar{a} \to \pm \sqrt{2a^2\omega/\omega \pm a}$$

Then for a lightly loaded cable or high forcing frequency

$$\bar{a} \to \pm\sqrt{2}\ a$$

and for a highly loaded cable or low forcing frequency

$$\bar{a} \to \sqrt{2a\omega}$$

Therefore, to insure damping at low values of λ_0 will always require the use of a light, heavily loaded cable.

Finite Cable - Fixed Ends

The finite cable with fixed ends may be obtained from the previously considered infinite cable by considering the cable, at an angle λ_0 to the airstream, to be clamped at two points along the x-axis, $x = 0$ and $x = \ell$. The equation to be solved is then the same as given in Equation II-109

$$\partial^2 z/\partial x^2 - \xi\ \partial z/\partial x - \xi\delta/V_0\ \partial z/\partial\tau = 1/a^2\ \partial^2 z/\partial t^2 \qquad \text{II-109}$$

with the boundary conditions

$$z(0,\ t) = z(\ell,\ t) = 0 \qquad \text{II-118}$$

and the initial conditions

$$z(x,\ 0) = \phi(x)$$
$$\partial z/\partial t\Big|_{t=0} = \Psi(x) \qquad \text{II-119}$$

Application of the method of separation of variables with the assumed solution

$$z(x,\ t) = \sum_{k=0}^{\infty} F_k(t)\ G_k(x) \qquad \text{II-120}$$

where the $G_k(x)$ are orthogonal, leads to the two equations

$$G''(x) - \xi\ G' + n^2 G = 0 \qquad \text{II-121a}$$

$$\ddot{F}(t) + a^2\xi\delta/V_0\ \dot{F}(t) + a^2 n^2 F = 0 \qquad \text{II-121b}$$

A solution of Equation II-121a after application of the boundary conditions of Equation II-118 is then

$$G_k(x) = e^{\xi/2\ x}\ A_k\ \sin k\pi/\ell\ x \qquad \text{II-122}$$

and $n^2 = k^2\pi^2/\ell^2 + (\xi/2)^2$.

Similarly, a solution of Equation II-121b is

$$F_k(t) = e^{-a^2\xi\delta/2V_0\ t}\ [B_k \cos a[k^2\pi^2/\ell^2 + \xi^2/4(1 - a^2\delta^2/4V_0^2)]^{1/2} t$$
$$+ C_k \sin a[k^2\pi^2/\ell^2 + \xi^2/4\ (1 - a^2\delta^2/4V_0^2)]^{1/2} t\ ] \qquad \text{II-123}$$

Substitution of Equations II-122 and 123 into Equation II-120, gives the result

$$z(x,\ t) = e^{(\xi/2\ x\ -\ a^2\xi\delta/2V_0\ t)}[D_k \cos a[k^2\pi^2/\ell^2 + \xi^2/4(1 - a^2\delta^2/4V_0^2)]^{1/2}t$$

$$+ E_k \sin a[k^2\pi^2/\ell^2 + \xi^2/4(1 - a^2\delta^2/4V_0^2)]^{1/2}t] \sin k\pi/\ell\ x \qquad \text{II-124}$$

Then since $G_k(x) = A_k\ e^{\xi/2\ x} \sin k\pi x/\ell$ is orthogonal, and A_k is simply a scale factor, it may be specified that

$$e^{\xi/2\ x} \sin k\pi x/\ell = H_k(x) \qquad \text{II-125}$$

where $H_k(x)$ is orthogonal.

Application of the initial conditions to Equation II-124 with Equation II-125 incorporated yields

$$\sum_{n=1}^{\infty} D_k\ H_k(x) = \Phi(x)$$

and

$$\sum_{n=1}^{\infty} [a[k^2\pi^2/\ell^2 + \xi^2/4(1 - a^2\delta^2/4V_0^2)]^{1/2}\ E_k -$$

$$\xi\delta a^2/2V_0\ D_k]\ H_k(x) = \Psi(x)$$

From which the coefficients D_k and E_k may be determined as

$$D_k = \frac{\int_0^\ell \Phi(x)\ H_k(x)\ dx}{\int_0^\ell H_k(x)^2\ dx} \qquad \text{II-126a}$$

$$E_k = \frac{\int_0^\ell \Psi(x)\ H_k(x)\ dx + \xi\delta a^2/2V_0 \int_0^\ell \Phi(x)\ H_k(x)\ dx}{a[k^2\pi^2/\ell^2 + \xi^2 4(1 - a^2\delta^2/4V_0^2)]^{1/2} \int_0^\ell H_k(x)^2 dx} \qquad \text{II-126b}$$

The solution of Equation II-109 with the initial and boundary conditions of Equations II-118 and 119 is then given by Equation II-124 with the coefficients evaluated from Equations II-126a and 126b.

Time Dependent Boundary Conditions

Consideration of the same time dependent boundary problem considered previously for a line in a vacuum, with the added aerodynamic

parameters, indicates the increase in complexity created by the aerodynamic terms. The equation of motion is given in Equation II-109. The boundary conditions are:

$$
\begin{aligned}
z(0,\ t) &= Ae^{-j\omega_0 t} = \beta_1(t) \\
z(\ell,\ t) &= Be^{-j\omega_\ell t} = \beta_2(t)
\end{aligned}
\qquad \text{II-127}
$$

and the initial condiiions are:

$$z(x,\ 0) = \partial z/\partial t \Big|_{t=0} = 0 \qquad \text{II-128}$$

The same procedure used in the previous time dependent boundary problem is followed. A solution

$$z(x,\ t) = y(x,\ t) + \sum_{i=1}^{2} \beta_i(t)\ \alpha_i(x) \qquad \text{II-128}$$

is assumed, where y(x, t) must satisfy the equation

$$
\begin{aligned}
a^2\ \partial^2 y/\partial x^2 - a^2\ \xi\ \partial y/\partial x - a^2\ \xi\delta/V_0\ \partial y/\partial t - \partial^2 y/\partial t^2 &= \sum_{i=1}^{2} -a^2\beta_i(t)\alpha_i''(x) \\
&+ a^2\ \xi\ \beta_i(t)\ \alpha_i'(x) + a^2\xi\delta/V_0\ \dot{\beta}_i(t)\ \alpha_i(x)\ddot{\beta}_i(t)\alpha_i(x)
\end{aligned}
\qquad \text{II-129}
$$

with the boundary conditions

$$
\begin{aligned}
y(0,\ t) &= \beta_1(t) - \sum_{i=1}^{2} \beta_i(t)\ \alpha_i(0) \\
y(\ell,\ t) &= \beta_2(t) - \sum_{i=1}^{2} \beta_i(t)\ \alpha_i(\ell)
\end{aligned}
\qquad \text{II-130}
$$

and the initial conditions

$$
\begin{aligned}
y(x,\ 0) &= -\sum_{i=1}^{2} \beta_i(0)\ \alpha_i(x) \\
\partial y/\partial t \Big|_{t=0} &= -\sum_{i=1}^{2} \dot{\beta}_i(0)\ \alpha_i(x)
\end{aligned}
\qquad \text{II-131}
$$

The $\alpha_i(x)$ are then chosen to make the right hand side of Equations II-130 vanish.

$$
\begin{aligned}
\alpha_1(x) &= \cos \omega_0/a\ x - \cot \omega_0/a\ \ell\ \sin \omega_0/a\ x \\
\alpha_2(x) &= \csc \omega_\ell/a\ \ell\ \sin \omega_\ell/a\ x
\end{aligned}
\qquad \text{II-132}
$$

Substitution into Equation II-129 yields the equation of motion in the form

$$a^2\ \partial^2 y/\partial x^2 - a^2\xi\ \partial y/\partial x - a^2\xi\delta/V_o\ \partial y/\partial t - \partial^2 y/\partial t^2 =$$

$$a\xi[-\omega_o A e^{-j\omega_o t}\ r_1(x) + \omega_\ell \csc \omega_\ell/a\ \ell\ e^{-j\omega_\ell t}\ r_2(x)] \qquad \text{II-133}$$

where

$$r_1(x) = (\cot \omega_o/a\ \ell + j\ a\delta/V_o)\cos \omega_o/a\ x + (1 - j\ a\delta/V_o\ \cot \omega_o/a\ \ell) \cdot \sin \omega_o/a\ x$$

and

$$r_2(x) = \cos \omega_\ell/a\ x - j\ a\delta/V_o\ \sin \omega_\ell/a\ x$$

Application of the method of separation of variables leads to an assumed solution of Equation II-133 of the form

$$y(x,\ t) = \sum_{n=1}^{\infty} X_n(x)\ T_n(t) \qquad \text{II-134}$$

and X_n is assumed to be an orthogonal function. The $r_1(x)$ and $r_2(x)$ may then be expanded in series

$$r_1(x) = \sum_{n=1}^{\infty} H_{1n}\ X_n$$

$$r_2(x) = \sum_{n=1}^{\infty} H_{2n}\ X_n \qquad \text{II-135}$$

where H_{1n} and H_{2n} are given by the orthogonality relations

$$H_{1n} = \int_0^\ell r_1(x)\ X_n\ dx / \int_0^\ell X_n^2\ dx$$

$$H_{2n} = \int_0^\ell r_2(x)\ X_n\ dx / \int_0^\ell X_n^2\ dx \qquad \text{II-136}$$

The boundary conditions, from Equations II-132 and 130, are

$$y(0,\ t) = y(\ell,\ t) = 0 \qquad \text{II-137}$$

and the initial conditions, from Equations II-131 and 132, are

$$y(x,\ 0) = -A(\cos \omega_0/a\ x - \cot \omega_0/a\ \ell \sin \omega_0/a\ x) - B \csc \omega_\ell/a\ \ell \cdot \sin \omega_\ell/a\ x \qquad \text{II-138}$$

$$\partial y/\partial t\Big|_{t=0} = Aj\omega_0(\cos \omega_0/a\ x - \cot \omega_0/a\ \ell \sin \omega_0/a\ x) + Bj\omega_\ell \csc \omega_\ell/a\ \ell \cdot \sin \omega_\ell/a\ x$$

Substitution of Equations II-134 and 135 into Equation II-133, separation of the variables, equating the space and time dependent portions to the constant $-\gamma_n^2$ and satisfying the boundary conditions yields the space dependent solution,

$$X_n = C_n e^{\xi/x} \sin n\pi/\ell\ x \qquad \text{II-139}$$

the auxillary equation

$$\gamma_n^2 = a^2[(n\pi/\ell)^2 + (\xi/2)^2]$$

and the time dependent solution

$$T_n = e^{\zeta_b t}\,[D_n \cos \sqrt{\gamma_n^2 - \zeta_b^2}\,t + E_n \sin \sqrt{\gamma_n^2 - \zeta_b}\ t] + T_p \qquad \text{II-140}$$

T_p is the particular integral resulting from the right hand side of Equation II-133 and $\zeta_b = a^2\xi\delta/2V_o$. T_p may be determined by variation of parameters to be

$$T_p = \frac{a\xi}{\sqrt{\alpha_n^2-\zeta_b^2}}\ \frac{\omega_0 A\, H_{1n}}{\gamma_n^2-\omega_0^2-2j\omega_0\zeta_b}\ [\sqrt{\gamma_n^2-\zeta_b^2}\ e^{-j\omega_0 t} - e^{-\zeta_b t}(\sqrt{\gamma_n^2-\zeta_b^2}\ \cdot$$

$$\cos\sqrt{\gamma_n^2-\zeta_b^2}\ t - (\zeta_b-j\omega_0)\sin\sqrt{\gamma_n^2-\zeta_b^2}\ t)] - \frac{\omega_\ell A H_{2n}}{\gamma_n^2-\omega_\ell^2-2j\omega_\ell\zeta_b}\ \cdot$$

$$[\sqrt{\gamma_n^2-\zeta_b^2}\ e^{-j\omega_\ell t} - e^{-\zeta_b t}(\sqrt{\gamma_n^2-\zeta_b^2}\cos\sqrt{\gamma_n^2-\zeta_b^2}\ t - (\zeta_b-j\omega_\ell)\ \cdot$$

$$\sin\sqrt{\gamma_n^2-\zeta_b^2}\ t)\ ] \qquad \text{II-141}$$

The parameters H_{1n} and H_{2n} may be obtained from Equations II-136. They are complex constants in γ_n, ω_ℓ, ζ_b and ξ and are very lengthy and are, therefore, not presented since they would only complicate the picture without adding any important parameters. They both contain the factor $1/B_n$

and the quantity $T_p^* = B_n T_p$ may therefore be defined without loss of generality. The solution of Equation II-133 may then be expressed

$$y(x, t) = \sum_{n=1}^{\infty} e^{(\xi/2\, x - \zeta_b t)} \left[F_n \cos \sqrt{\gamma_n^2 - \zeta_b^2}\, t + G_n \sin \sqrt{\gamma_n^2 - \zeta_b}\, t \right] \cdot$$

$$\sin n\pi/\ell\, x + T_p^* e^{\xi/2\, x} \sin n\pi/\ell\, x \qquad \text{II-142}$$

where the constants F_n and G_n are obtained by satisfying the initial conditions. The final form of $z(x, t)$ could then be obtained by a substitution into Equation II-128.

The example considered here is that of a cable forced at both ends. Energy is supplied to both ends of the cable at constant frequencies and the resulting wave form is that of a standing wave. The waves, however, are damped with both time and distance upstream and amplified with distance downstream. If the standing wave is considered as a series of direct and reflected traveling waves, then all waves introduced at the downstream end, whether direct or reflected, are damped. However, waves introduced at the upstream end may be damped or undamped depending on the distance of the point in question from the initial end and the wave velocity. The time required for the damping to override the amplification at any given point along the cable is then

$$t_d = V_o x / a^2 \delta$$

and the primary items of importance are again the freestream velocity, the angle of the cable with the stream, and the tension to mass ratio of the cable as discussed previously.

The end conditions representing the case of a towed vehicle are somewhat different than in the example considered. The end conditions

for the towed vehicle would actually be the dynamic equations of motion of the system. Energy is then both added and removed from the cable during different periods of time. The motion of vehicles of this type are approximately sinusoidal and the case just considered would be a fairly good approximation of the energy input contriubtion. However, the energy removed from the string disturbs the reflexion relations in both amplitude and phase. Consequently, cancellation of the downwind wave does not occur in the same manner. If the energy inputs and removals occurred at the same frequency, a standing wave simular to the example would exist. However, they are not and the resulting wave form is that of a traveling wave superposed on a standing wave.

III EQUATIONS OF MOTION OF A TOWED BODY

The equations of motion of a towed body are essentially the same as the dynamic equations expressing the motion of a free body such as an airplane. The primary difference, aside from possible configuration differences, is the pull of the tow cable which varies with the dynamic motion of the towed body.

This section will briefly describe the general development of the dynamic equations of the towed vehicle and develop useful representations of the cable force.

General Development

The general development of the basic equations of motion of a vehicle in air are well described in References 8, 9, and 10. Consequently only a brief outline will be presented here.

Figure 10 demonstrates the configuration considered. The x, y, z axis system is fixed at the center of gravity of the body. The X, Y, Z axes are fixed in space. Application of Newton's second law then gives the vector equations of motion

$$\bar{F} = m\, d\bar{V}/dt \qquad \text{III-1}$$

$$\bar{M} = d\bar{h}/dt \qquad \text{III-2}$$

where $\bar{F}$ and $\bar{M}$ are the external applied forces and moments, $\bar{V}$ is the velocity of the center of gravity of the body of mass m, and $\bar{h}$ is the total moment of momentum of the body. These equations are referred to the X, Y, Z axes fixed in space. If these axes are considered to be fixed at the center of gravity of the vehicle and to move with it, the reference frame is rotating and the equations become

$$\bar{F} = m\,\partial\bar{V}/\partial t + m\,\bar{\omega} \times \bar{V}c \qquad \text{III-3}$$

$$\bar{M} = \partial\bar{h}/\partial t + \bar{\omega} \times \bar{h} \qquad \text{III-4}$$

Resolution of each of these total vectors into its components, along and about the axes, gives the Eulerian equations of motion as:

$$\begin{aligned}
F_x &= m(\dot{U} + QW - RV) \\
F_y &= m(\dot{V} + RU - PW) \\
F_z &= m(\dot{W} + PV - QU) \\
L &= \dot{P}I_x - \dot{R}I_{xz} + QR(I_z - I_y) - PQI_{xz} \\
M &= \dot{Q}I_y + PR(I_x - I_z) - R^2 I_{xz} + P^2 I_{xz} \\
N &= \dot{R}I_z - \dot{P}I_{xz} + PQ(I_y - I_x) + QRI_{xz}
\end{aligned} \qquad \text{III-5}$$

where F_x, F_y, and F_z are the externally applied forces along the x, y, z axes; L, M, and N are the externally applied moments about the x, y, and z axes; U, V, and W are vehicle velocities along the x, y, and z axes; P, Q, and R are rotational velocities about the x, y, and z axes; I_x, I_y, I_z are moments of inertia about the respective axes; I_{xz} is the product of inertia about the y-axis, and the xz plane is taken as a plane of symmetry. Equations III-5 are then the general non-linear equations of motion.

Linearization

The equations of motion may be linearized by the assumption that all perturbations away from the equilibrium state will be small. The velocities of Equations III-5 may then be taken as summations of the steady state values, indicated by the subscript $_o$, plus the perturbation quantities, indicated by lower case letters.

$$
\begin{aligned}
U &= U_o + u \\
V &= V_o + v \\
W &= W_o + w \\
P &= P_o + p \\
Q &= Q_o + q \\
R &= R_o + r
\end{aligned}
\qquad \text{III-6}
$$

Substitution of Equations III-6 into Equations III-5, and the neglection of products and squares of perturbation quantities gives, with the assumption of wing level steady flight with all velocity components zero except U_o and W_o,:

$$
\begin{aligned}
F_x &= m[\dot{u} + W_o q] \\
F_y &= m[\dot{v} + U_o r - W_o p] \\
F_z &= m[\dot{w} - U_o q] \\
L &= \dot{p} I_x - \dot{r} I_{xz} \\
M &= \dot{q} I_y \\
N &= \dot{r} I_z - \dot{p} I_{xz}
\end{aligned}
\qquad \text{III-7}
$$

The Euler axes are assumed to rotate from one position to another in the order ψ, θ, φ as shown in Figure II. The relation between the angular velocities and these Euler angles, which are small since the perturbations are small, are then

$$
\begin{aligned}
p &= \dot{\varphi} \\
q &= \dot{\theta} \\
r &= \dot{\psi}
\end{aligned}
\qquad \text{III-8}
$$

<u>Gravity and Aerodynamic Forces</u>

The gravity force on a body in the atmosphere acts along a line through the center of the earth. If the equilibrium Euler axes are

displaced form this line by the angles θ_o and φ_o, then the steady flight components of the weight force are given by

$$g_{x_o} = W \sin \theta_o$$

$$g_{y_o} = 0 \qquad \text{III-9}$$

$$g_{z_o} = W \cos \theta_o \cos \varphi_o$$

where the wing level steady flight condition $\varphi_o = 0$ has been incorporated. In disturbed flight, these components must be rotated through the Euler angles ψ, θ, φ. This gives the weight components along the disturbed axes, with the disturbance assumed small, as:

$$g_x = - W \sin \theta_o - W \cos \theta_o \ \theta$$

$$g_y = W \sin \theta_o \ \psi + W \cos \theta_o \ \varphi \qquad \text{III-10}$$

$$g_z = - W \sin \theta_o \ \theta + W \cos \theta_o$$

The aerodynamic forces and moments acting on the vehicle may be expanded in a Taylor's series about an equilibrium position as follows

$$G = G_o + (\bar{G}_\alpha)_o \ \alpha + (\bar{G}_\beta)_o \ \beta + (\bar{G}_u)_o \ u + \ldots + \text{H.O.T.}$$

where $(\bar{G}_\alpha)_o = \partial G/\partial\alpha$ evaluated at the equilibrium conditions, etc. and H. O.T. indicates higher order terms. The α, β, u, etc. are the parameters with which the aerodynamic forces and moments are known to vary. They are perturbation quantities and the higher order terms which consist of powers of the quantities of order two or greater may be neglected. If X, Y, and Z are used to indicate the aerodynamic forces along the x, y, and z axes and L, M, and N are used to indicate the moments about the axes, the expansions of the aerodynamic forces and moments in terms of parameters with which each is known to vary in quasi-steady flow are then

$$X = X_o + \bar{X}_u u + \bar{X}_q q + \bar{X}_w w + \bar{X}_{\dot{w}} \dot{w} + \bar{X}_{\dot{u}} \dot{u} + \bar{X}_{\dot{q}} \dot{q} \qquad \text{III-11a}$$

$$Y = Y_o + \bar{Y}_r r + \bar{Y}_v v + \bar{Y}_p p + \bar{Y}_{\dot{r}} \dot{r} + \bar{Y}_{\dot{v}} \dot{v} + \bar{Y}_{\dot{p}} \dot{p} \qquad \text{III-11b}$$

$$Z = Z_o + \bar{Z}_u u + \bar{Z}_q q + \bar{Z}_w w + \bar{Z}_{\dot{u}}\dot{u} + \bar{Z}_{\dot{q}}\dot{q} + \bar{Z}_{\dot{w}}\dot{w} \qquad \text{III-11c}$$

$$L = L_o + \bar{L}_r r + \bar{L}_v v + \bar{L}_p p + \bar{L}_{\dot{r}}\dot{r} + \bar{L}_{\dot{v}}\dot{v} + \bar{L}_{\dot{p}}\dot{p} \qquad \text{III-11d}$$

$$M = M_o + \bar{M}_u u + \bar{M}_q q + \bar{M}_w w + \bar{M}_{\dot{u}}\dot{u} + \bar{M}_{\dot{q}}\dot{q} + \bar{M}_{\dot{w}}\dot{w} \qquad \text{III-11e}$$

$$N = N_o + \bar{N}_v v + \bar{N}_{\dot{v}}\dot{v} + \bar{N}_r r + \bar{N}_{\dot{r}}\dot{r} + \bar{N}_p p + \bar{N}_{\dot{p}}\dot{p} + \bar{N}_w w \qquad \text{III-11f}$$

Then, if the Eulerian axis system is specified as a stability axis system, $W_o = 0$ and all partial derivatives with respect to rates of change of velocities are zero. The equations of motion obtained from Equations III-11, 10 and 7 become:

$$m\dot{u} = X_o + \bar{X}_u u + \bar{X}_q q + \bar{X}_w w - W \sin \gamma_o - W \cos \gamma_o \theta + \bar{T}_x z + T_{x_o}$$

$$m(\dot{v} + U_o r) = Y_o + \bar{Y}_r r + \bar{Y}_v v + \bar{Y}_p p + W \sin \gamma_o \psi + W \cos \gamma_o \varphi + \bar{T}_y z$$

$$m(\dot{w} - U_o q) = Z_o + \bar{Z}_u u + \bar{Z}_q q + \bar{Z}_w w - W \sin \gamma_o \theta + W \cos \gamma_o + \bar{T}_z z + T_{z_o}$$

$$\dot{p} I_x - \dot{r} I_{xz} = L_o + \bar{L}_r r + \bar{L}_v v + \bar{L}_p p + \bar{T}_{L} z \qquad \text{III-12}$$

$$\dot{q} I_y = M_o + \bar{M}_u u + \bar{M}_q q + \bar{M}_w w + \bar{T}_M z + T_{M_o}$$

$$\dot{r} I_z - \dot{p} I_{xz} = N_o + \bar{N}_v v + \bar{N}_r r + \bar{N}_p p + \bar{N}_w w + \bar{T}_N z$$

where the $\bar{T}_{()}$ terms represent variable force parameters and the $T_{()_o}$ steady state cable force values.

For the system in equilibrium flight, all of the perturbation terms and time derivatives are zero. The equilibrium equations are then

$$X_o - W \sin \gamma_o + T_{x_o} = 0$$

$$Z_o + W \cos \gamma_o + T_{z_o} = 0 \qquad \text{III-13}$$

$$M_o + T_{M_o} = 0$$

$$Y_o = L_o = N_o = 0$$

Substitution of Equations III-13 into Equations III-12; division of the first three equations of Equation III-12 by m, and division of the roll equation by I_x, the pitch equation by I_y and the yaw equation by I_z; and substitution of Equations III-8 yields the final form of the equations of motion.

$$\dot{u} - X_u u - X_q\dot{\theta} - X_w w + g\theta \cos \gamma_o - T_x z = 0$$
$$\dot{w} - U_o\dot{\theta} - Z_u u - Z_q\dot{\theta} - Z_w w + g\theta \sin \gamma_o - T_z z = 0 \qquad \text{III-14}$$
$$\ddot{\theta} - M_u u - M_q\dot{\theta} - M_w w - T_M z = 0$$

$$\dot{v} + U_o\dot{\psi} - Y_n\dot{\psi} - Y_v v - Y_p\dot{\varphi} - \psi g \sin \gamma - g\varphi \cos \gamma_o - T_y = 0$$
$$\ddot{\varphi} - \ddot{\psi}\,(I_{xz}/I_x) - L_r\dot{\psi} - L_v v - L_p\dot{\varphi} - T_L = 0 \qquad \text{III-15}$$
$$\ddot{\psi} - \ddot{\varphi}\,(I_{xz}/I_z) - N_r\ddot{\psi} - N_v v - N_p\dot{\varphi} - N_w w - T_N = 0$$

The parameters in Equations III-14 and III-15 obtained by dividing the partial derivatives of the aerodynamic forces by inertial terms (i.e. $X_u = 1/m\,\bar{X}_u$; $M_u = 1/I_y\,\bar{M}_u$; $L_v = 1/I_x\,\bar{L}_r$; $N_r = 1/I_z\,\bar{N}_r$; etc.) are termed dimensional stability derivatives.

The aerodynamic forces and moments may be expressed in the form

$$F = C_F\ 1/2\ \rho V^2\ S$$

and

$$M = C_M\ 1/2\ \rho V^2\ S\ C$$

where C_F and C_M are dimensionless coefficients, ρ is the fluid density, V is the total vehicle velocity and S and C are characteristic area and length. Utilizing these definitions in obtaining the partial derivatives, the dimensional stability derivatives of Equations III-14 and III-15 may be obtained. A complete derivation of these terms may be found in Reference 10. The dimensional stability derivatives of interest here are presented in Table I.

Cable Forces

The transverse vibrations of the cable were seen in Section II, for the case of time varying boundaries, to vary with both time and position along the cable. The cable form can then be considered to consist of a traveling wave superposed upon a standing wave. In particular, since

TABLE I

DIMENSIONAL STABILITY DERIVATIVES

Longitudinal

$$X_u = (\rho US/m)(-C_{D_u} - C_D)$$

$$X_w = (\rho US/2m)(C_L - C_{D_\alpha})$$

$$X_q \approx 0$$

$$Z_u = (\rho US/m)(C_{L_u} - C_L)$$

$$Z_w = (\rho US/2m)(-C_{L_\alpha} - C_D)$$

$$Z_q = (\rho US\bar{c}/4m)\ C_{L_q}$$

$$M_u = (\rho US\bar{c}/I_y)(C_{m_n} + C_m)$$

$$M_w = (\rho US\bar{c}/2I_y)\ C_{m_\alpha}$$

$$M_q = (\rho US\bar{c}^2/4I_y)\ C_{m_q}$$

Lateral

$$Y_v = (\rho US\ /2m)\ C_{y_\beta}$$

$$Y_r = (\rho USb/4m)\ C_{y_r}$$

$$Y_p = (\rho USb/4m)\ C_{y_p}$$

$$L_v = (\rho USb/2I_x)\ C_{\ell_\beta}$$

$$L_r = (\rho USb^2/4I_x)\ C_{\ell_r}$$

$$L_p = (\rho USb^2/4I_x)\ C_{\ell_p}$$

$$N_v = (\rho USb/2I_z)\ C_{n_\beta}$$

$$N_r = (\rho USb^2/4I_z)\ C_{n_r}$$

$$N_w = (\rho USb/2I_z)\ C_{n_\alpha}$$

over 80 percent of the energy in a fixed end vibrating cable is contained in the fundamental mode, the cable form may be considered as a traveling wave superposed on the fundamental which in effect is the equilibrium cable shape. In addition, the traveling wave has been shown to be damped in both the upstream and downstream direction at low velocities which are considered in this report. Consequently, the traveling wave may be neglected and the cable may be assumed to vary from one equilibrium shape to another with perturbations in the towed vehicle position.

Therefore, to determine the cable contributions of the dynamic stability of the system, the following assumptions are made:

1) The cable shape is that of its fundamental mode and remains that way through out all reasonable perturbations from the equilibrium.

2) Small angle approximations are valid.

The cable tension T may be resolved into the horizontal and vertical components:

$$T \cos \lambda \qquad \text{III-16}$$

and

$$T \sin \lambda \qquad \text{III-17}$$

respectively. The equations of motion are such that the variation of these tension components for a given disturbance must be known(i.e. $dT \cos \lambda/dz$, etc. must be determined). Differentiation of Equation III-16 with respect to z gives:

$$\frac{d\,T \cos \lambda}{dz} = -T_0 \sin \lambda_0 \frac{d\lambda}{dz} + \cos \lambda \frac{dT}{dz}$$

dT/dz and $dT/d\lambda$ are determined as follows. The derivative, $dT/d\lambda$, is obtained from Figure 8 by measuring the slope of the point corresponding to the known values of φ, the equilibrium cable angle, α, the cable parameter

and the ratio T_o/T. The expression $d\lambda/d\eta$ may be obtained from Equation II-101 by first replacing φ by the identification $\varphi = 90 - \lambda$ and after simple algebraic manipulation, differentiation with respect to η. Multiplication of $dT/d\lambda$ by $d\lambda/d\eta$ yields $dT/d\eta$ which is the non-dimensional form of dT/dz.

It is noted, however, that this development did not account for any changes in cable contributions with a change in length of the cable. This can be seen from Figure 7; for although the cable becomes shorter, the attachment point remains fixed and thus the value of the cable derivatives remain constant. Therefore, for a given displacement Δz, the change in λ is approximately a function of ℓ as follows (see Figure 12):

$$\frac{d\lambda}{dz} \approx \frac{\sec\lambda}{\ell} \cdot \cos\lambda \approx \frac{1}{\ell} \qquad \text{III-18}$$

This effect was computed for a range of α's of 2° through 20° and a number of rope lengths. The results of this addition to the $d\varphi/dz$ term previously developed are discussed in a later section and are most significant.

The cable contributions in the longitudinal equations, Equations III-14, with the perturbation angle α neglected in comparison with the large angle λ are:

$$T_x = 1/m[\cos\lambda_o(\partial T/\partial z) - T_o \sin\lambda_o(\partial\lambda/\partial z)]$$

$$T_z = 1/m[\sin\lambda_o(\partial T/\partial z) + T_o \cos\lambda_o(\partial\lambda/\partial z)] \qquad \text{III-19}$$

$$T_m = 1/I_y[e(T_o \cos\lambda_o(\partial\lambda/\partial z) + \sin\lambda_o(\partial T/\partial z)) - a(\cos\lambda_o(\partial T/\partial z) - T_o \sin\lambda_o(\partial\lambda/\partial z))$$

The geometric parameters b and a may be determined from Figure 13.

The terms T_Y, T_L, and T_N of Equations III-15 could be determined from a consideration of the variation of the cable shape that results from a lateral displacement in the same manner as the longitudinal case was

developed. However, the contributions due to this variation in cable shape is small, since the displacements to be considered are small, and in addition they are stabilizing contributions. Consequently, the effect of neglecting changes in cable shape will be minor and conservative.

The cable is therefore assumed to remain straight in the lateral directional planes. The lateral directional cable forces may then be determined from the geometry of Figures 14 and 15, with all angles assumed to be small, to be

$$
\begin{aligned}
T_Y &= 1/m[(1 + e/\ell)\beta - y^*/\ell - (1 + a/n)\varphi][T_o + (\partial T/\partial z)z] \\
T_L &= a/I_x[(1 + e/\ell)\beta - y^*/\ell - (1 + a/n)\varphi][T_o + (\partial T/\partial z)z] \qquad \text{III-20} \\
T_N &= b/I_x[(1 + e/\ell)\beta - y^*/\ell - (1 + a/n)\varphi][T_o + (\partial T/\partial z)z]
\end{aligned}
$$

The complete equations of motion may then be obtained when necessary by incorporating Equations III-19 and 20 into Equations III-14 and 15 respectively. There are then six equations with eight time dependent variables. This is, of course, an undesirable situation. It may be alleviated, however, by considering the rate of side displacement and the rate of climb relations. The rate of climb $\dot{z}$ is given by

$$\dot{z} = U \sin \gamma$$

where γ is the flight path angle. The small perturbations assumption and the assumption that γ_o is very small or zero leads to

$$\dot{z} = U_o \gamma$$

Integration and the substitution of $\gamma = \theta - \alpha$ then gives the desired form

$$z = \int_{t_o}^{t} U_o(\theta - \alpha)\, d\tau \qquad \text{III-21}$$

Likewise, the rate of side displacement is given by

$$\dot{y}^* = U_o \psi$$

where y^* is the displacement perpindicular to the equilibrium x-axis.

Upon integration, this gives

$$y^* = \int_{t_0}^{t} U_0 \, \psi \, d\tau \qquad \text{III-22}$$

With the addition of Equations III-21 and 22, the equations are in a complete form which may be solved.

IV LONGITUDINAL DYNAMICS

The study of the longitudinal dynamics of a towed vehicle proceeded in the normal manner from simplified analytical studies to the more complicated studies which required the use of an analog computer. The simplified studies were designed to indicate important vehicle parameters and to determine the value of different degrees of approximations of the cable tension variation.

In order to establish relative magnitude of the various parameters, it was necessary to specialize the studies to a particular basic configuration. The basic configuration chosen is shown in Figure 13. The static stability derivatives for this configuration were obtained from preliminary wind tunnel data of tests performed at the David Taylor Model Basin. The equilibrium flight parameters were calculated from Equations III-13. The procedure followed is described in Appendix B and the parameters for several conditions are listed. The basic flight condition of 170 m.p.h. at sea level and a cable length of 250 feet were chosen for the majority of the studies.

The results are presented in the form of stability boundaries where applicable, and the effects of variations in the other parameters are discussed.

Two Degree of Freedom Studies

The two degree of freedom studies were performed with the assumptions that both the tension in the cable and the angle between the cable and the body centerline ($\bar{\lambda}$) remain constant. The cable contributions of Equations III-14 are then

$$T_x z = (-T_0/m \sin\lambda_0)\alpha = -T_x^*$$

$$T_z z = (T_0/m \cos\lambda_0)\alpha = T_z^*$$

$$T_m z = T_0/I_y(e\cos\lambda_0 + a\sin\lambda_0)\alpha = T_m^*$$

With the flight path angle γ_0 taken as zero, and $w = U_0\alpha$, the Laplace transformed equations of longitudinal motion become:

$$\begin{aligned} &X_q s\theta + (X_u - s)u + (X_\alpha - T_x^*)\alpha = 0 \\ &(Z_q + U_0)s\theta + Z_u u - (U_0 s - Z_\alpha + T_z^*)\alpha = 0 \qquad \text{IV-1} \\ &-(s^2 - M_q s)\theta + M_u u + M_\alpha + T_M^* = 0 \end{aligned}$$

The two fundamental modes of motion of most vehicles of this type are the short period and the phugoid. The short period is highly damped and the velocity of the vehicle changes very little before its effect becomes negligible. Therefore, the short period mode may be studied approximately by setting $u = 0$ and studying the lift and moment equations. Setting $\gamma_0 = 0$, the characteristic equation for the short period mode is then:

$$\begin{vmatrix} (Z_q + U_0)s & -U_0 s^2 + (Z_\alpha - T_z^*) \\ -(s^2 - M_q s) & M_\alpha + T_M^* \end{vmatrix} = 0 \qquad \text{IV-2}$$

Expanding, this becomes:

$$[s^2 - [M_q + (Z_\alpha/U_0)] - T_z^*/U_0]s + [(M_q/U_0)(Z_\alpha - T_z^*) - (Z_q/U_0)(M_\alpha + T_M^*) - M_\alpha - T_M^*] = 0 \qquad \text{IV-3}$$

Equation IV-3 is of the form $S^2 + 2\zeta\omega_n s + \omega_n^2 = 0$, and we know that for stability $2\zeta\omega_n = -[M_q + Z_\alpha/U_0 - T_z^*/U_0]$ must be positive. Therefore, the criterion for stability of this approximation to the short period mode is:

$$[M_q + (Z_\alpha/U_0) - (T_z^*/U_0) < 0 \qquad \text{IV-4}$$

An order of magnitude investigation of the parameters involved indicates that $M_q < 0$, $Z_\alpha/U_0 < 0$ and $T_z^*/U_0 > 0$. Therefore, $[M_q + Z_\alpha/U_0 - T_z^*/U_0]$ < 0 and the approximation to the short period oscillation is stable.

The motion in the Phugoid mode consists mainly of altitude and speed changes with very small α variations. This mode may then be studied by setting $\alpha = 0$. In addition, the motion is slow so that the inertia terms are negligible and since M_q and M_u are usually very small, the pitching moment equation may be neglected. The characteristic equation of the Phugoid mode is then:

$$\begin{vmatrix} X_q s - g & X_u - s \\ (Z_q + U_o)s & Z_u \end{vmatrix} = 0 \qquad \text{IV-5}$$

where again $\gamma_o = 0$. Expansion then yields:

$$s^2 + \left(\frac{Z_u X_q}{U_o + Z_q} - X_u \right) s - \frac{g Z_u}{U_o + Z_q} = 0 \qquad \text{IV-6}$$

The criterion for Phugoid stability is then that

$$[(Z_u X_q / U_o + Z_q) - X_u] > 0 \qquad \text{IV-7}$$

An order of magnitude investigation indicates:

$$X_u < 0 \qquad X_q \leq 0$$

$$Z_u < 0 \qquad Z_q \leq 0$$

$$U_o + Z_q > 0$$

Therefore, $[Z_u X_q / (U_o + Z_q) - X_u] > 0$ and the <u>approximation</u> to the phugoid motion is stable.

<u>Three Degree of Freedom Analytical Studies</u>

The dependence of the phugoid approximation upon the parameters $M_q = 0$ and $M_u = 0$ makes it desirable to check the complete three-degree of freedom stability for this configuration. To check the importance of the q derivatives, a typical flight condition of 170 mph and an aerodynamic center location of 2.5 feet aft of the center of gravity have been used. A one degree change in the effective angle of attack due to a positive q

rotation about the c.g. requires the creation of a vertical velocity of 4.37 feet/sec. at the aerodynamic center. The vertical velocity at the aerodynamic center due to $\bar{q}$ is given by

$$w = \bar{q}(x_{ac} - x_{cg})$$

Solving this equation for $\bar{q}$, it is seen that a pitch velocity of 100 deg/sec is required to give a one degree change in effective angle of attack. Therefore, it may be assumed that $C_{L_q} = C_{D_q} = C_{M_q} = 0$, and correspondingly, $Z_q = X_q = M_q = 0$.

The dimensional stability derivative M_u is given by:

$$M_u = (\rho U_0 S\bar{c}/I_{yy})(C_{M_u} + C_{M_0})$$

The coefficient C_{M_u} is a Mach number effect and may of course be assumed zero in the speed range of interest. C_{M_0} is the aerodynamic coefficient of pitching moment in the equilibrium flight condition and may be suspected to have an appreciable negative value to balance the moment created by the line tension. The value of C_{M_0} is obtained from the appendix as $C_{M_0} = -.28$ which as anticipated is an appreciable negative value and the three degree of freedom stability must be checked.

Constant Tension - Constant $\bar{\lambda}$

The three degree of freedom characteristic equations corresponding to the assumption of constant tension and constant $\bar{\lambda}$ is obtained from Equations IV-1 as:

$$\begin{vmatrix} X_q s & X_u - s & X_\alpha - T_x* \\ (Z_q + U_0)s & Z_u & -(U_0 s - Z_\alpha + T_z*) \\ -(s^2 + M_q s) & M_u & M_\alpha + T_M* \end{vmatrix} = 0 \qquad \text{IV-8}$$

Expanding the determinant yields the equation in the form

$$As^4 + Bs^3 + Cs^2 + Ds + E = 0 \qquad \text{IV-9}$$

where:

$$A = 1$$

$$B = -[X_u + (Z_\alpha/U_o) - (T_z{*}/U_o)]$$

$$C = [X_u(Z_\alpha/U_o - T_z{*}/U_o) - M_\alpha - Z_u(X_\alpha/U_o - T_x{*}/U_o) - T_M{*}]$$

$$D = [M_u(g - X_\alpha + T_x{*}) + X_u(M_\alpha + T_M{*})]$$

$$E = g[Z_u(M_\alpha/U_o + T_M{*}/U_o) - M_u(Z_\alpha/U_o - T_z{*}/U_o)]$$

With the equation in this form, the stability may be checked by applying Routh's stability criteria for a quartic. Routh's criteria for a quartic state that for stability, the constants of the equation must meet the conditions that

$$A > 0, \quad B > 0, \quad D > 0, \quad E > 0$$

and

$$D(BC - AD) - B^2E > 0$$

To apply these criteria to equation IV-9, the dimensional stability derivatives must be evaluated numerically. For the equilibrium conditions stated previously, these values are:

$$X_u = -0.197; \qquad Z_u = -0.094$$

$$M_u = -0.052; \qquad X_\alpha/U_o = +0.047$$

$$Z_\alpha/U_o = -1.978; \qquad M_\alpha/U_o = -0.78$$

$$T_x/U_o = +\underline{0.0818}; \qquad T_z/U_o = +\underline{0.098}$$

$$T_M/U_o = +\underline{0.0456}$$

The coefficients of Equation IV-9 may be evaluated as:

$$A = +1 > 0$$

$$B = +2.28 > 0$$

$$C = +129 > 0$$

$$D = +22 > 0$$

$$E = -1.39 < 0$$

and it may be seen that the first part of Routh's criteria is not satisfied for $E < 0$. Checking the second part,

$$D\ (BC - AD) - B^2E = +\ 7577 > 0$$

and the criterion is satisfied.

One of Routh's criteria is not satisfied and the system is therefore unstable. However, the second criterion is satisfied and it may therefore be suspected that the instability is weak. This type of instability is characterized by the depletion of the phugoid mode from an oscillation into two real roots, one of which is convergent and the other divergent. This phenomenon is here directly due to the strong M_u term. Ordinarily, in piloted or otherwize controlled aircraft, this divergence resulting from the M_u term would cause no great concern, for it is normally a slow divergence and can readily be controlled. In this case, however, control is not possible and the effects of further variations in the cable contribution must be considered. If necessary, this effect may be alleviated by shifting the center of gravity forward or by otherwize changing the configuration so that the equilibrium position moment due to line tension is reduced.

Tension Constant, $\bar{\lambda}$ Semi-constant

The next step toward the development of the actual system analysis is to again consider that T remains constant at its equilibrium value, but that the angle $\bar{\lambda}$ is semi-constant. What is meant by considering $\bar{\lambda}$ as a semi-constant can best be seen by considering the vehicle at its initial displaced position following a disturbance. According to the basic no line harmonics assumption for these studies mentioned previously, the line would now assume a tension and an angle λ with the relative wind that would

agree with some equilibrium vehicle configuration at that position in space. The tension is now assumed constant at the original equilibrium value T_0. The angle λ at this initial displaced position now has some value $\lambda_0 + \lambda_1$. Then since $\lambda = \alpha + \bar{\lambda}$, $\lambda_1 = \alpha_1 + \bar{\lambda}_1$. The variation in λ from λ_1 is then assumed to consist only of α variations, and $\bar{\lambda}$ remains constant at $\bar{\lambda} = \bar{\lambda}_1$, such that λ at any later time is given by $\lambda = \alpha + \bar{\lambda}_1$, or $\lambda = \alpha + \bar{\lambda}$ where $\bar{\lambda}$ is considered constant. This then allows transfer functions to be established in much the same way as control deflection transfer functions are established.

The equations of motion with this assumption and $X_q = Z_q = M_q = \gamma_0 = 0$ are then:

$$\begin{aligned}
&- g\theta + X_u u - \dot{u} + X_\alpha \alpha - T_x^* \alpha = T_x^* \lambda \\
&U_0 \dot{\theta} + Z_u u + Z_\alpha \alpha - U_0 \dot{\alpha} - T_z^* \alpha = T_z^* \lambda \\
&\ddot{\theta} - M_u u - M_\alpha \alpha - T_M^* \alpha = T_M^* \lambda
\end{aligned} \qquad \text{IV-10}$$

Laplace transforming, these equations become:

$$\begin{matrix} -g & X_u - s & X_\alpha - T_x^* \\ U_0 s & Z_u & Z_\alpha - T_z^* - U_0 s \\ s^2 & -M_u & -(M_\alpha + T_M^*) \end{matrix} \quad \begin{matrix} \theta \\ u \\ \alpha \end{matrix} \quad = \quad \begin{matrix} T_x \\ T_z \\ T_M \end{matrix} \quad \lambda \qquad \text{IV-11}$$

The transfer functions of interest may now be defined as:

$$\frac{\theta(s)}{\lambda(s)} = \frac{N_0}{\Delta}\,; \qquad \frac{U(s)}{\lambda(s)} = \frac{N_u}{\Delta}\,; \qquad \frac{\alpha(s)}{\lambda(s)} = \frac{N_\alpha}{\Delta}$$

The denominator is then:

$$\begin{aligned}
\Delta = U_0 \;\; s^4 &- [(Z_\alpha - T_z^*/U_0) + X_u]\, s^3 + [X_u(Z_\alpha - T_z^*/U_0) - Z_u(X_\alpha - T_x^*/U_0) \\
&- M_\alpha - T_M^*]\, s^2 + [X_u(M_\alpha + T_M^*) + M_u(g - X_\alpha + T_x^*)]\, s + g[Z_u(M_\alpha + T_M^*/U_0) \\
&+ M_u(Z_\alpha - T_z^*/U_0)]
\end{aligned} \qquad \text{IV-12}$$

$$N_\theta = U_0\, T_M^* \quad s^2 - [(T_X^*/T_M)M_u + (T_Z^*/T_M)(M_\alpha/U_0) + X_u + (Z_\alpha/U_0)]\, s$$
$$+ T_X^*/T_M(M_u(Z_\alpha/U_0) - Z_u(M_\alpha/U_0)) + T_Z^*/T_M(X_u(M_\alpha/U_0) - M_u(X_\alpha/U_0))$$
$$+ X_u(Z_\alpha/U_0) - X_u(X_\alpha/U_0)$$

$$N_u = -U_0\, T_X^* \quad s^3 + [(T_Z^*/T_X^*)(X_\alpha/U_0) - (Z_\alpha/U_0)]\, s^2 + [(T_M^*/T_X^*)\cdot$$
$$(g - X_\alpha) - M_\alpha]s - g[(T_Z/T_X)(M_\alpha/U_0) + (T_M/T_X)(Z_\alpha/U_0)] \qquad \text{IV-13}$$

$$N_\alpha = -T_Z \quad s^3 - [X_u - (T_X/T_Z)Z_u + U_0\, T_M]\, s^2 + [U_0(T_X/T_Z)M_u$$
$$+ U_0(T_M/T_Z)X_u]s + g[M_u + Z_u(T_M/T_Z)]$$

Substituting the values of the dimensional stability derivatives presented previously, the transfer functions are:

$$\frac{\theta(s)}{\lambda(s)} = \frac{(11.4)(s + .209)(s + 3.735)}{(s + .2185)(s - .0315)(s^2 + 2.083\, s + 183.65)}$$

$$\frac{U(s)}{\lambda(s)} = \frac{-20.45(s + .318)(s^2 + 1.717\, s + 206.454)}{(s + .2185)(s - .0315)(s^2 + 2.083\, s + 183.65)} \qquad \text{IV-14}$$

$$\frac{\alpha(s)}{\lambda(s)} = \frac{-0.0982(s - 2849.89)(s^2 + .0119\, s + .0011)}{(s + .2183)(s - .0313)(s^2 + 2.003\, s + 183.65)}$$

Approximate zero-pole, s-plane locations of these transfer functions are shown in Figure 16. The system will, of course, operate at points determined by the gain of the transfer functions. The gain in each case is seen, from Equations IV-13, to be a direct function of the line tension. In view of this dependence on line tension and the fact that allowing $\bar{\lambda}$ to be semi-constant does not alter the location of the pole in the right half plane appreciably, direct attention was focused on the variable tension case.

Static Effect of Variable Tension

Essentially, the tension in the line will be expressed by an expansion about the equilibrium tension. This expression has the basic form:

$$T = T_o + \partial T/\partial\lambda \; \lambda \qquad \text{IV-15}$$

Some insight as to the expected results may, however, be obtained by looking at the static stability contribution of the $\partial T/\partial\lambda$ term which is known to be negative.

Consider a situation where the vehicle is disturbed by an initial change in angle of attack α at some time t_1. Let the subscript $_o$ indicate equilibrium conditions and the subscript $_1$ the conditions at time t_1. Referring to Figure 17, the moment about the c.g. due to the line tension is positive and is given by:

$$M_T = a\,T \sin\beta \qquad \text{IV-16}$$

Since M_T is positive, the condition for static stability is:

$$M_{T_1} < M_{T_o} \qquad \text{IV-17}$$

The change in position of the vehicle that results from the change in α is not being considered (this would be a dynamics study), only the conditions at the time the change in α is applied. Therefore, $\bar{\lambda}_1 = \bar{\lambda}_o$ and since $\bar{\lambda} = \beta + \xi$, where ξ is a constant dependent only on the geometry of the vehicle,

$$\sin\beta_1 = \sin\beta_o \qquad \text{IV-18}$$

From equation IV-16, we have for M_{T_1} and M_{T_o}

$$M_{T_1} = T_1 \; a \sin\beta_1 \qquad \text{IV-19}$$

$$M_{T_o} = T_o \; a \sin\beta_o \qquad \text{IV-20}$$

Then dividing

$$\frac{M_{T_1}}{M_{T_o}} = \frac{T_1 \; a \sin\beta_1}{T_o \; a \sin\beta_o} = \frac{T_1}{T_o} \qquad \text{IV-21}$$

Now from the basic relation, equation IV-15:

$$T_1 = T_o + \partial T/\partial\lambda \; \lambda$$

Now $$\lambda = \Lambda_1 - \Lambda_o \qquad \text{IV-22}$$

and $$\Lambda_1 = \bar{\lambda}_1 + \alpha_1 = \bar{\lambda}_1 + \alpha_o + \alpha \qquad \text{IV-23}$$

$$\Lambda_1 = \bar{\lambda}_o + \alpha_o \qquad \text{IV-24}$$

Then substituting Equations IV-23 and 24 in Equation IV-22

$$\lambda = \bar{\lambda}_1 + \alpha - \bar{\lambda}_o,$$

and since $\bar{\lambda}_1 = \bar{\lambda}_o$

$$\lambda = \alpha > 0 \qquad \text{IV-25}$$

Then since $\partial T/\partial\lambda < 0$

$$\partial T/\partial\lambda \; \lambda < 0 \qquad \text{IV-26}$$

and $$T_o + \partial T/\partial\lambda < T_o \qquad \text{IV-27}$$

Therefore, from Equations IV-15 and IV-27, it is seen that

$$T_1 < T_o \qquad \text{IV-28}$$

and from Equations IV-21 and IV-28,

$$M_{T_1}/M_{T_o} = T_1/T_o < 1$$

Consequently, relation IV-17

$$M_{T_1} < M_{T_o}$$

is satisfied and the line tension contribution is statically stabilizing. It appears, therefore, at least from static stability considerations, the inclusion of variable line tension will stabilize the system.

Complete Three Degree of Freedom Studies

The foregoing analyses have indicated that the basic system may be expected to be dynamically stable when the complete effect of the cable forces are included in the analysis. This may of course be done analytically; however, the ultimate aim is a study of the variations of several of the parameters away from their basic values. Therefore, an analog computer was utilized for the remainder of the study.

The longitudinal equations of Equations III-14 may be put in a form better suited to a parametric study on an analog computer. This may be accomplished by substitutions of the non-dimensional stability derivative forms of Table I for the dimensional stability derivatives in the equations and defining the operator D = d/dt. The Equations III-14 then become

$$[D + (\rho US/m)C_{D_0}]u - (\rho U^2S/2m(C_L - C_{D_\alpha})\alpha + g\theta - T_X z = 0$$
$$(\rho US/m)C_L u + [U_0 D + (\rho US/2m)(C_{L_\alpha} + C_{D_0})]\alpha - DU_0\theta - T_Z z = 0 \qquad \text{IV-29}$$
$$(\rho US\bar{c}/I_y)C_{M_0} - (\rho U^2 S\bar{c}/2I_y)C_{M_\alpha}\alpha - [D^2 - (\rho US\bar{c}^2/4I_y)C_{M_q} D]\theta - T_M z = 0$$

Equations IV-29 along with Equations III-19 and III-21 were solved on the analog computer to obtain the results discussed in the remainder of this section.

All of the aerodynamic parameters of Equations IV-29 were varied to some degree. The three parameters of primary importance were determined to be C_{M_α}, C_{M_q} and C_{M_0}. The first two, C_{M_α} and C_{M_q} are normally important in common airplane dynamics and the effect was as suspected. The parameter C_{M_0} is somewhat peculiar to towed vehicles since it has an unusually high negative value as discussed in an earlier section.

The changes in the other parameters in general have very little individual effect. In particular, any variation that results from either C_{D_0} or C_{L_0} changes can be traced directly to the C_{M_0} term mentioned above. The parameters C_{L_α} and C_{D_α} were varied independently of C_{M_α} in order to divorce the effect. Variation in the parameter C_{L_α} were investigated separately and in combination with variations in C_{D_α}. Increases in C_{L_α} alone produced mild decreases in the phugoid frequencies and accompanying

increases in damping. In itself this is not particularly important but it is a peculiarity of the present configuration that C_{D_α} is exactly zero; thus, it was felt that a combined variance of the two parameters was required. C_{D_α} variations had no effect on the characteristic response for a large number of C_{L_α} values.

The parameter C_{m_q} was assumed to be negligible in the preliminary analyses because of the high pitch rate required to make it effective. Further investigation revealed, however, that a damping term was present due to a cable whiplash effect following a displacement of the vehicle. The damping occurs because the energy required to start the cable oscillation, which of course is damped since it moves upstream along the cable, is removed from the vehicle oscillation. Therefore, a term $C_{m_{d\theta}}$ was included in the computer analyses and very effectively damps the short period oscillation. The effect of $C_{m_{d\theta}}$ on the phugoid is best described by reference to the stability boundary of Figure 18. From the figure, it is seen that the minimum allowable value of C_{m_α} for stability decreases as $C_{m_{d\theta}}$ increases. This effect is quite similar to the effect of C_{m_q} in normal aircraft.

The effect of C_{m_α} variations on the frequency of the phugoid motion was significant. The detailed derivation of this parameter is included in the appendix; only note for this discussion that the large values obtained are due to the particular configuration. C_{m_α} is essentially the restoring force of the system but also appears as a contributing factor in the damping of the phugoid mode. Increasing C_{m_α} increases the damping but decreases the frequency; the opposite is true for decreases in C_{m_α}. Along with these changes a rather significant change in altitude variation

for a given input was noted, an increase of C_{m_α} increasing the amplitude of the altitude variation. The vehicle can not be compared to a simple spring mass damper case for illustrative purposes; the system is simply too complex. C_{m_α} contributes to both the restoring force and damping terms with the latter effect being the more significant.

The cable parameter C_{m_C} is not actually a stability derivative in the normal sense of the word. C_{m_C} and C_{m_0} are directly related; as one increases so does the other and visa versa. This may be seen simply by considering the equation of static equilibrium. As discussed previously, however, even though C_{m_0} is not a stability derivative, it plays a major role in determining the dynamic response of the vehicle, thus the inclusion of a stability boundary depending upon C_{m_0} as Figure 19.

V LATERAL DYNAMICS

The study of the lateral-directional dynamics proceeded in the same way as the longitudinal study except that exclusive use was make of the analog computer. The study was again specialized to the basic configuration of Figure 13 and the same basic flight condition of 170 mph at sea level and a cable length of 250 feet was utilized. The dynamic stability derivatives were estimated for the basic flight condition. The estimation procedures are described in Appendix A.

The results are again presented in the form of stability boundaries of the important parameters.

Three Degree of Freedom Studies

These studies are restricted to the lateral-directional modes of motion. Therefore, the terms in Equation III-20 which are functions of z and the N_w term of Equations III-15 are omitted. In other words, the tension in the cable is assumed to remain constant throughout the lateral oscillations and the body is assumed to be perfectly symmetrical about its centerline. Then substituting the approximation

$$v \approx U_0 \beta$$

and Equations III-20 and III-22 into Equations III-15 after replacing the dimensional stability derivatives with the non-dimensional derivatives of Table I, the lateral-directional equations of motion become:

$$[D - (\rho US/2m)C_{y_\beta} - (T_0/mU_0)\ell_T]\beta - [(\rho Sb/4m)C_{y_p}D + (g/U_0) - (T_0/mU_0)\ell_n]\varphi + [(D^2/U_0)(1 - (\rho Sb/4m)C_{y_r}) + (T_0/mU_0\ell)]y^* = 0 \quad \text{V-1a}$$

$$- [(\rho U^2 Sb/2I_x)C_{\ell_\beta} + (a/I_x)\ell_T T_o]\beta + [D^2 - (\rho USb^2/4I_x)C_{\ell_p}D + (a/I_x) \cdot T_o\ell_n]\varphi - [((I_{xz}/I_x)D^3 + (\rho USb^2/4I_x)C_{\ell_r}D^2)(1/U_o) - (a/I_x)(T_o/\ell)]y^* = 0 \qquad \text{V-1b}$$

$$- [(\rho U^2 Sb/2I_z)C_{n_\beta} + (e/I_z)\ell_T T_o]\beta - [(I_{xz}/I_x)D^2 + (\rho USb^2/4I_z)C_{n_p}D - (e/I_z)T_o\ell_n]\varphi + [D^3 - (\rho USb^2/4I_z)C_{n_r}D^2)(1/U_o) + (T_o/I_z)(\ell_T - 1)]y^* = 0 \qquad \text{V-1c}$$

where $\ell_T = 1 + e/\ell$; $\ell_n = 1 + a/n$.

Equations V-1 were solved on the analog computer to obtain the results that follow.

The lateral equations were investigated in the same manner as the longitudinal; first variations of individual parameters then combined variations. In this mode there are long period and short period oscillations. The dutch roll oscillation, representing the short period mode in the lateral case, was never found to be unstable and in fact little could be done to effect the damping. The mode of real concern is not the dutch roll, however, but rather the pendulum mode becuase it is the mode which couples with its counterpart, the longitudinal phugoid, to produce an unstable oscillatory motion. Thus, the following remarks concerning parameter variations will be oriented towards their effects on the pendulum mode.

Individual variations of C_{n_β}, C_{n_p}, C_{ℓ_r}, C_{ℓ_p} and C_{y_r} produce no appreciable effect. Decreasing either T_o or C_{n_r} decreases the damping and frequency of the pendulum mode while increasing either has the opposite effects. The combined variation of the parameters C_{n_β} and T_o has a negligible effect. However, decreases in both C_{n_p} and C_{y_β} do result in a

measurable decrease in damping and frequency. But no instabilities were produced by magnitude variations of these parameters within the limits of the computer.

Two combinations did produce instabilities. These were variations of C_{n_β} with C_{ℓ_β} and C_{n_β} with C_{n_r}. Stability boundaries were determined for these variations and are presented in Figures 20 and 21. From Figure 20 it is seen that the boundary is quite similar to one that might be obtained for a normal airplane. The oscillation represented is, of course, the pendulum oscillation and not the dutch roll for which no instabilities were found. Figure 21 indicates that below a certain amount of yaw damping, there are restrictions on the minimum value of C_{n_β} for a stable system. All of these parameters are dependent on the size of the vertical tail and stability will in general be increased by an increase in vertical tail size. However, the location of the added area with respect to the roll axis must be carefully considered so that the value of C_{ℓ_β} will not increase to the instability point.

VI LONGITUDINAL-LATERAL COUPLED DYNAMICS

Following the investigation of the longitudinal and lateral modes separately, the equations were coupled by adding the z dependent and N_w terms of Equations III-20 and III-15 to Equations V-1 and solving the resulting equations simultaneously with the longitudinal Equations IV-29, III-19 and III-21.

The same procedure of single variation, combined variation and finally coupled variations of the significant parameters was followed in this investigation. Because of the nature of the coupling, variations of lateral parameters produced no changes in longitudinal dynamics. The opposite was not true, however. For example, decreasing $C_{m_{d\theta}}$, which decreased the longitudinal short period damping, had a noticeable effect on the dutch roll oscillation of the lateral mode. This is to be expected because the excitation of the dutch roll is from the yaw moment produced by the α variation.

Probably the most significant fact to come to light in this investigation is the existance of a stray coupling between the phugoid and pendulum modes. Aspects of this coupling are discussed in the following sections.

Effect of C_{n_α} and Variable Cable Lengths

Observations made of the vehicle in flight indicated that the period of the phugoid was an inverse function of the cable length. For very short cable lengths, i.e. five feet or less, the vehicle developed an unstable oscillation coupling the longitudinal and lateral modes of motion. The analog investigation revealed that the coupled oscillations were those of the longitudinal phugoid and lateral pendulum modes.

The necessary coupling term was found from wind tunnel data to be a yawing moment due to change in angle of attack. This seems to arise from the use of a drag skirt to obtain high drag coefficients. The resulting region of separated flow is in all probability sensitive to angle of attack changes, thus providing the condition of asymmetry producing C_{n_α}.

As was discussed in the section on Cable Dynamics, analytical studies indicated that as the cable became shorter, the phugoid frequency increased. An investigation of the coupled equations of motion on the analog computer bore out this prediction in full. Further, as the plots of φ and z (Figure 22), which represent the pendulum and phugoid modes respectively, indicate the coupled oscillations do indeed develop and increase in intensity as the cable becomes shorter.

Increases in phugoid frequency aggravate the coupled motion discussed above. The most significant parameter in this respect is, of course, rope length, although others do exist. Decreasing C_{m_α} and the combination of decreasing C_{m_α} and simultaneously increasing C_{m_0} have relatively strong influences with respect to increasing the phugoid frequency. In the lateral mode, any increase in frequency of the pendulum mode was accompanied by an increase in damping with the net result being neglibible.

Besides configuration change's of the vehicle, cable characteristics were varied to determine if they might have any influence on this aspect of the study. As can be seen from Figure 23, a plot of phugoid period vs. length, the effect of varying α^o, essentially the weight to cross section ratio of the cable, decreases rapidly with the smaller values of ℓ. Thus, it is felt that α^o is relatively unimportant.

REFERENCES

1. Sokolnikoff, I. S. and R. M. Redheffer, Mathematics of Physics and Modern Engineering, McGraw Hill, New York, 1958.

2. Coulson, C. A., Waves, Interscience Publishers, Inc., New York, 1961.

3. Bland, D. R., Vibrating Strings, The Free Press, Glencoe, Illinois, 1960.

4. Mindlin, R. D. and L. E. Goodman, "Beam Vibrations with Time-Dependent Boundary Conditions," Journal of Applied Mechanics, Vol. 17, 1950.. pp. 377.

5. Glauert, M. A., The Stability of a Body Towed by a Light Wire, Reports and Memoranda No. 1312, Gr. Britain Air Ministry Aeronautical Research Committee, February, 1930.

6. Glauert, M. A., Heavy Flexible Cable for Towing a Heavy Body Below an Aeroplane, Reports and Memorands No. 1592, Gr. Britain Air Ministry Aeronautical Research Committee, February, 1934.

7. Pode, L., Tables for Computing the Equilibrium Configuration of a Flexible Cable in a Uniform Stream, David Taylor Model Basin Report 687, March, 1951.

8. Etkin, B., Dynamics of Flight, John Wiley & Sons, Inc., New York, 1959.

9. Perkins, C. D., and R. E. Hage, Airplane Performance Stability and Control, John Wiley & Sons, Inc., New York, 1949.

10. BuAer, Fundamentals of Design of Piloted Aircraft Flight Control Systems, Volume II, Dynamics of the Airframe, BuAer Rept. AE-61-4, February, 1953.

11. Hoerner, S. F., Fluid Dynamic Drag, Published by the author, Midland Park, New Jersey, 1958.

APPENDIX A

Derivation of Stability Derivatives

The longitudinal stability derivatives were obtained from the first wind tunnel report from DTMB; however, several of the lateral stability derivatives were calculated analytically, a discussion of the technique follows.

C_{n_r}:

Let the distance from the c.g. to the aerodynamic center of the vertical stabilizer be e. The yaw velocity, r, produces a side velocity at the tail.

$$V_T = er$$

The side slip angle at the tail is then,

$$\beta_T = - V_T/U_0 = - er/U_0 \sim \text{radians}$$

The lift curve slope of the vertical tail is taken as $C_{L_{\beta_T}}$, then defining $C_{y_{\beta_T}} = C_{L_{\beta_T}} S_T/S$, the total force due to sideslip becomes:

$$F_T = (C_{y_{\beta_T}} \beta_T)(1/2\ \rho U^2 S)$$

The yawing moment due to this force is,

$$N = - eF_T$$

Therefore,

$$C_N = \frac{+ e\ C_{y_{\beta_T}} \cdot er/U_0 \cdot 1/2\ \rho U^2 S}{1/2\ \rho U^2 Sb}$$

and after cancellation and multiplication by b/b

$$C_N = (2e^2/b^2) C_{y_{\beta_T}} \cdot rb/2U$$

Differentiating this expression with respect to rb/2U, yields C_{n_r} as:

$$C_{n_r} = \frac{\partial C_n}{\partial \frac{rb}{2U}} = 2(e/b)^2 C_{y_{\beta_T}}$$

The coefficient C_{y_r} may be obtained in much the same manner. From the equation,

$$C_y = C_{y_{\beta_T}}(-er/U_o) = -(2e/b)\, C_{y_{\beta_T}}\ (rb/2U_o)$$

C_{y_r} is obtained by differentiating this expression for C_y with respect to $rb/2U_o$, thus,

$$C_{y_r} = -(2e/b)C_{y_{\beta_T}}$$

C_{ℓ_r}:

(See Figure A1)

Defining $v_r = f_r$

$$\alpha = w/U = \alpha_o + \Delta\alpha$$

$$U_L = U_o + U_r$$

$$U_R = U_o - U_r$$

$$\alpha_o = w/U_o$$

therefore, $w/U - w/U_o = \Delta\alpha$

The incremental changes in angle of attack on the left and right wing are respectively,

$$\Delta\alpha_L = -\alpha_o\ (U_r/U_o + U_r)$$

and

$$\Delta\alpha_R = +\alpha_o\ (U_r/U_o - U_r)$$

From the foregoing analysis, ΔL_R takes the following form

$$\Delta L_r = L_r - L_o = 1/2\ \rho U_r^2\ S\ C_{L_\alpha}\ \alpha_r - 1/2\ \rho U_o^2\ S\ C_{L_\alpha}\ \alpha_o$$

which becomes

$$\Delta L_r = 1/2\ \rho S\ C_{L_\alpha}\ [(U_o - U_r)^2(\alpha_o + \Delta\alpha_r) - (U_o^2\alpha_o)]$$

This is further reduced to the final form of

$$\Delta L_r = 1/2\ \rho S\ C_{L_\alpha}\ (-U_oU_r\alpha_o)$$

The same procedure was followed in obtaining ΔL_L,

$$\Delta L_L = 1/2\ \rho S\ C_{L_\alpha}\ U_o U_r \alpha_o$$

Defining ΔL as being positive in the negative z direction, the expression for the roll moment ℓ is:

$$\ell = -\ g\Delta L_r + g\Delta L_L$$

which after substitution and cancellation reduces to,

$$\ell = +\ g\rho S C_{L_\alpha}\ U_o U_r \alpha_o.$$

where g is the distance from the centerline to the M.A.C.

Thus, C_ℓ becomes,

$$C_\ell = (4g/b^2)\ C_{L_\alpha}\ \alpha_o\ \ (U_r b/2U_o)$$

since

$$U_r = v_r \cos \Omega = rf \cos \Omega$$

and $f = f \cos \Omega$

Upon substitution, U_r becomes:

$$U_r = gr$$

Thus, in final form, C_ℓ is

$$C_\ell = (2g/b)^2\ C_{L_\alpha}\ \alpha_o\ \ (rb/2U_o)$$

By differentiating with respect to $rb/2U_o$ and making the substitution of $\alpha_o{}^* = \alpha_o - \alpha_{o_L}$, C_{ℓ_r} is,

$$C_{\ell_r} = \left(\frac{g}{b/2}\right)^2 C_{L_\alpha}\ \alpha_o{}^*$$

C_{y_p}:

Due to the symmetry of the vertical stabilizers about the x axis, the contribution of these surfaces is negligible. Also, due to the small perturbation assumption, the contribution of the horizontal surfaces may be taken as zero. Therefore,

$$C_{y_p} = 0$$

C_{ℓ_p}:

C_{ℓ_p} has been estimated by calculating the average angle of attack at the m.a.c. resulting from the roll velocity p. (See Figure 2A.) Again,

$$\ell = - g\Delta L_r + g\Delta L_l$$

where

$$\Delta L_r = C_{L_\alpha} \alpha_r \, gS; \quad \Delta L_l = C_{L_\alpha} \alpha_l \, qS$$

and

$$\alpha_r \approx + gp/U_o; \quad \alpha_l \approx - gp/U_o$$

Thus, upon substitution, ℓ becomes,

$$\ell = - 2g \, C_{L_\alpha} \, gS \, gp/U_o$$

which, when nondimensionalized, is:

$$C_\ell = - \left(\frac{g}{b/2} \right)^2 C_{L_\alpha} \, pb/2U_o$$

By differentiating C_ℓ with respect to $pb/2U_o$, C_{ℓ_p} is found to have the form,

$$C_{\ell_p} = - \left(\frac{g}{b/2} \right)^2 C_{L_\alpha}$$

C_{n_p}:

The vertical fins again contribute nothing to this parameter because of their symmetrical location. The wing contribution stems from two sources; the change in drag due to the new angle of attack resulting from roll and the moment produced as a result of the non-parallel alignment of the lift vectors.

Because only small perturbations from the equilibrium position where C_{D_α} is zero are considered, the contribution from the first source is negligible.

Establishing the positive value of ΔD as pointing in the negative x direction the expression for the yaw moment N, becomes, (See Figure A3)

$$N = g\Delta D_r - g\Delta D_L$$

From Figure 4A, it may be seen that

$$\Delta D_r = - L_r \sin \Delta\alpha_r \approx - L_r \Delta\alpha_r$$

and

$$\Delta D_L = - L_L \sin \Delta\alpha_L = - L_L \Delta\alpha_L$$

The expression for L_L is as follows,

$$L_L = 1/2 \rho S U_L{}^2 C_{L_L} = 1/2 \rho S U_L{}^2 C_{L_\alpha} \alpha_L$$

also

$$L_r = 1/2 \rho S U_r{}^2 C_{L_\alpha} \alpha_r$$

By making the identification,

$$\alpha = \alpha_o + \Delta\alpha = (w_o/U_o) + (w/U)$$

and substituting the expression for α into the previously developed terms for L_r and L_L, they become,

$$L_r = 1/2 \rho S U_r{}^2 C_{L_\alpha}[(w_o/U_o) + (w_r/U_r)]$$

and

$$L_L = 1/2 \rho S U_r{}^2 C_{L_\alpha}[(w_o/U_o) + (w_L/U_L)]$$

Then with $w_L = - gp$ and $w_r = + gp$, the expressions for ΔD_r and ΔD_L become,

$$\Delta D_r = - 1/2 \rho S U_r{}^2 C_{L_\alpha}[\alpha_o + (gp/U_r)](gp/U_r)$$

and

$$\Delta D_L = - 1/2 \rho S U_L{}^2 C_{L_\alpha}[\alpha_o - (gp/U_L)](- gp/U_r)$$

Using the identities,

$$U_L{}^2 = U_o{}^2 + w_L{}^2; \quad U_r{}^2 = U_o{}^2 + w_r{}^2$$

and the inequalities,

$$w_r{}^2 \ll U_o{}^2; \qquad w_L{}^2 \ll U_o{}^2$$

the expression for N becomes,

$$N = 1/2\ \rho S g\ [2U_o C_{L_\alpha}\ \alpha_o\ gp]$$

Then,

$$C_n = -(4g^2/b^2)\ C_{L_\alpha}\ \alpha_o\ (pb/2U_o)$$

Differentiating with respect to $pb/2U_o$, the desired derivative is obtained as:

$$C_{n_p} = -\frac{g^2}{(b/2)^2}\ C_{L\alpha}\ \alpha_o$$

The values of the longitudinal and lateral stability derivatives for the basic configuration at a velocity of 170 mph at sea leval are presented in Table A-1.

TABLE A-1

NONDIMENSIONAL STABILITY DERIVATIVES

BASIC CONFIGURATION

$C_{y_\beta} = -\ 1.605$	$C_{\ell_r} = +\ 0.021$
$C_{n_\beta} = +\ 0.647$	$C_{y_p} = 0$
$C_{\ell_\beta} = -\ 0.067$	$C_{n_p} = -\ 0.021$
$C_{y_r} = +\ 2.86$	$C_{\ell_p} = -\ 0.83$
$C_{n_r} = -\ 2.37$	$C_{m_\alpha} = -\ 6.88$
$C_{m_q} = -\ 0.076$	$C_{L_\alpha} = 3.58$

APPENDIX B

Calculation of Equilibrium Flight Conditions

The equilibrium equations, for the case of zero flight path angle, are obtained from Equations III-13 as:

$$X_o + T_{X_o} = 0$$
$$Z_o + W + T_{Z_o} = 0 \qquad \text{B-1}$$
$$M_o + T_{M_o} = 0$$

The cable tension components in these equations are:

$$T_{X_o} = T_o \cos \lambda_o$$
$$T_{Z_o} = -T_o \sin \lambda_o \qquad \text{B-2}$$
$$T_{M_o} = -T_o[e \sin(\alpha_o - \lambda_o) + a \cos(\alpha_o - \lambda_o)]$$

The aerodynamic terms are then

$$X_o = -1/2\ \rho U^2 S[C_{D_{MIN}}\ (C_{L_\alpha}{}^2(\alpha_o - \alpha_{o_L})/\pi A)]$$
$$Z_o = -1/2\ \rho U^2 S\ C_{L_\alpha}(\alpha_o - \alpha_{o_L}) \qquad \text{B-3}$$
$$M_o = 1/2\ \rho U^2 Sc[C_{M_{o_L}} + C_{M_\alpha}\ (\alpha_o - \alpha_{o_L})]$$

where the subscript o_L indicates the zero lift condition. Substitution of Equations B-2 and B-3 into Equation B-1 and assuming $\alpha_o \ll \lambda_o$, yields the equations:

$$T_o \cos \lambda_o - 1/2\ \rho U^2 S[C_{D_{MIN}} \frac{C_{L_\alpha}{}^2\ (\alpha_o - \alpha_{o_L})^2}{\pi A}] = 0 \qquad \text{B-4}$$

$$T_o \sin \lambda_o - W + 1/2\ \rho U^2 S\ C_{L_\alpha}\ (\alpha_o - \alpha_{o_L}) = 0 \qquad \text{B-5}$$

$$1/2\ \rho U^2 Sc\ (C_{M_{o_L}} - C_{M_\alpha}\alpha_{o_L}) + e\ T_o \sin \lambda_o - aT_o \cos \lambda_o - [T_o(a \cos \lambda_o$$
$$+ b \sin \lambda_o) - 1/2\ \rho U^2 Sc\ C_{M_\alpha}]\alpha_o = 0 \qquad \text{B-6}$$

These three equations must now be solved for the unknown factors T_o, λ_o and α_o.

Substitution of Equations B-4 and B-5 into B-6 then gives the cubic equation for α_o

$$\alpha_o^3 - (2\alpha_{o_L} + (a\pi A/eC_{L_\alpha}) - a/e)\alpha_o^2 + [(C_{D_{MIN}}\pi A/C_{L_\alpha}^2) + \alpha_{o_L}^2 + (W\pi A/1/2\ \rho U^2 S C_{L_\alpha}^2)\ a/e + (a\pi A/eC_{L_\alpha})\ \alpha_{o_L} - \bar{c}/e\ (C_{m_\alpha}\pi A/C_{L_\alpha}^2) + \pi A/C_{L_\alpha} - 2\ a/e\ \alpha_{o_L}]\alpha_o - (W\pi A/1/2\ \rho U^2 S C_{L_\alpha}^2) - (\pi A/C_{L_\alpha})\alpha_{o_L} + a/e\ (C_{D_{MIN}}\pi A/C_{L_\alpha}^2) + a/e\ \alpha_{o_L}^2 - (\bar{c}\pi A/eC_{L_\alpha}^2)(C_{M_{o_L}} - C_{M_\alpha}\alpha_{o_L}) = 0 \quad \text{B-7}$$

Equation B-7 is then solved numerically for α_o.

With α_o known, Equations B-4 and B-5 are then solved for λ_o

$$\lambda_o = \tan^{-1} \frac{W + qSC_{L_\alpha}\alpha_{o_L} - qSC_{L_\alpha}\alpha_o}{q\bar{S}[C_{D_{MIN}} + (C_{L_\alpha}^2/\pi A)(\alpha_{o_L}^2 + \alpha_o^2 - 2\alpha_{o_L}\alpha_o)} \quad \text{B-8}$$

The cable tension is then obtained from Equation B-5.

$$T_o = \frac{W - qSC_{L_\alpha}(\alpha_o - \alpha_{o_L})}{\sin \lambda_o} \quad \text{B-9}$$

TABLE B-1

U	α_o	λ_o	T_o	alt.
150	-0.94	60.25	70.0	SL
170	-1.35	41.65	95.5	SL
200	-1.96	33.95	112.4	SL
250	-2.45	26.60	181.5	SL
300	-2.77	21.90	252.0	SL
350	-2.94	19.35	345.0	SL
170	-1.29	43.20	93.0	2000'
170	-1.10	53.00	78.1	5000'
250	-2.42	27.55	172.7	2000'

APPENDIX C

Computer Analysis

Early in the investigation, it became apparent that the problem would lend itself nicely to analysis by analog computer techniques. The computer work progressed concurrently with analytical studies following the standard approach of first examining the uncoupled equations of motion and then a complete, i.e. coupled, set of equations.

The longitudinal equations are really only three in number, the highest order being second. What may appear as a fourth equation is the integral expression for z which, when multiplied by the proper coefficients, becomes the cable contribution to the system dynamics. Electronic multipliers were used since they are standard equipment on the Pace TR-10 computers used for this work.

The lateral equations were three in number, the highest order being third.

The coupled equations were six in number, and are essentially the same equations as were treated independently, the only significant change being the addition of the C_{n_α} term in the lateral equation. This is discussed in another section. The coupling required the addition of three more multipliers and a number of amplifiers and pots. When completely coupled, three units were slaved employing some 52 operational amplifiers, a tribute to the flexibility and capability of the Pace equipment.

Figure C-1 is a schematic of the wiring diagram used in the computer study.

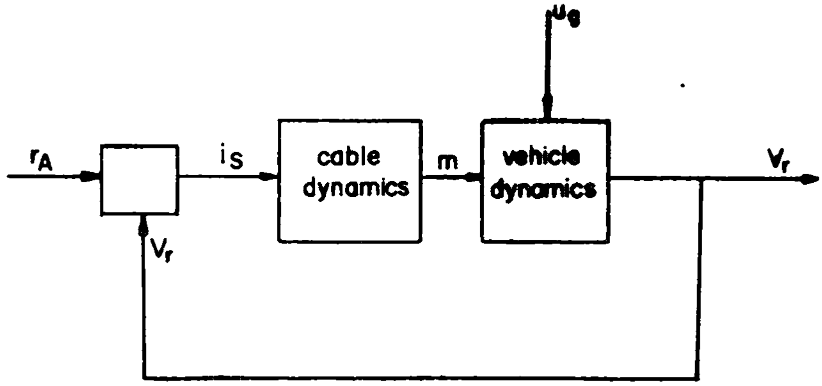


FIGURE 1. COMPLETE SYSTEM REPRESENTATION

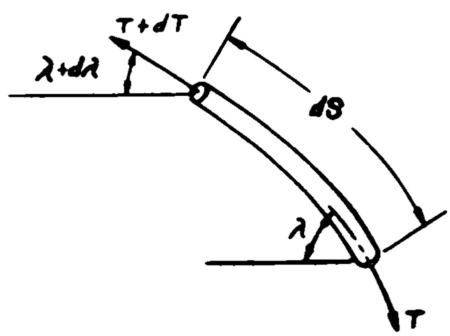


FIGURE 2. DISTURBED CABLE ELEMENT IN VACUUM

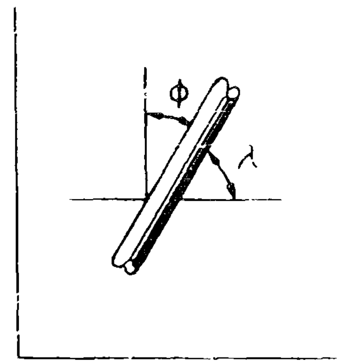


FIGURE 3. CABLE ELEMENT IN EQUILIBRIUM

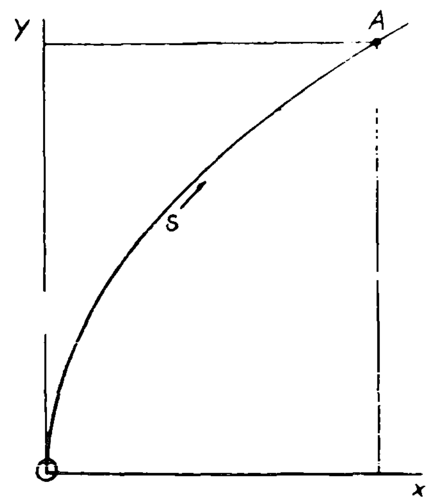


FIGURE 4. CONFIGURATION OF CABLE SUPPORTING A ZERO DRAG BODY

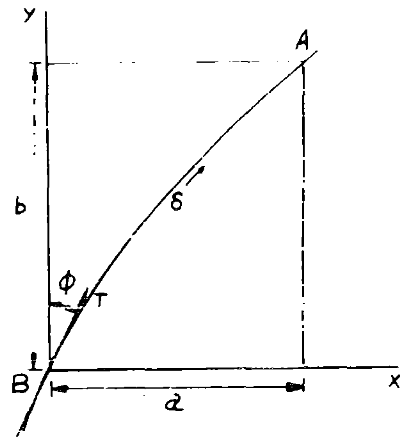


FIGURE 5. CONFIGURATION OF CABLE SUPPORTING A BODY WITH DRAG

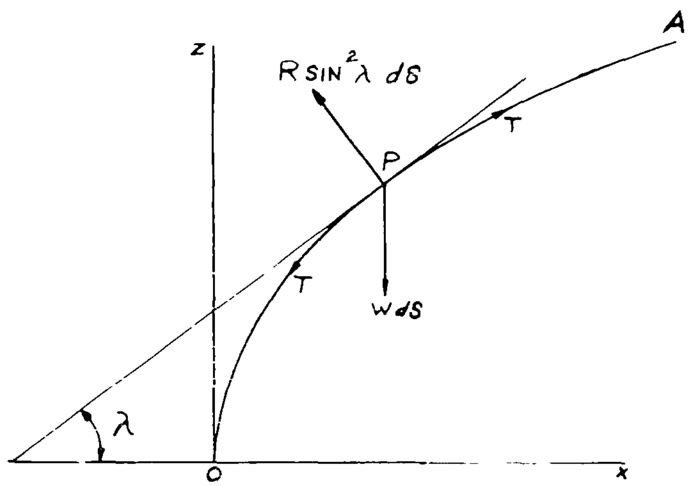


FIGURE 6. FORCES ON A CABLE IN A UNIFORM STREAM

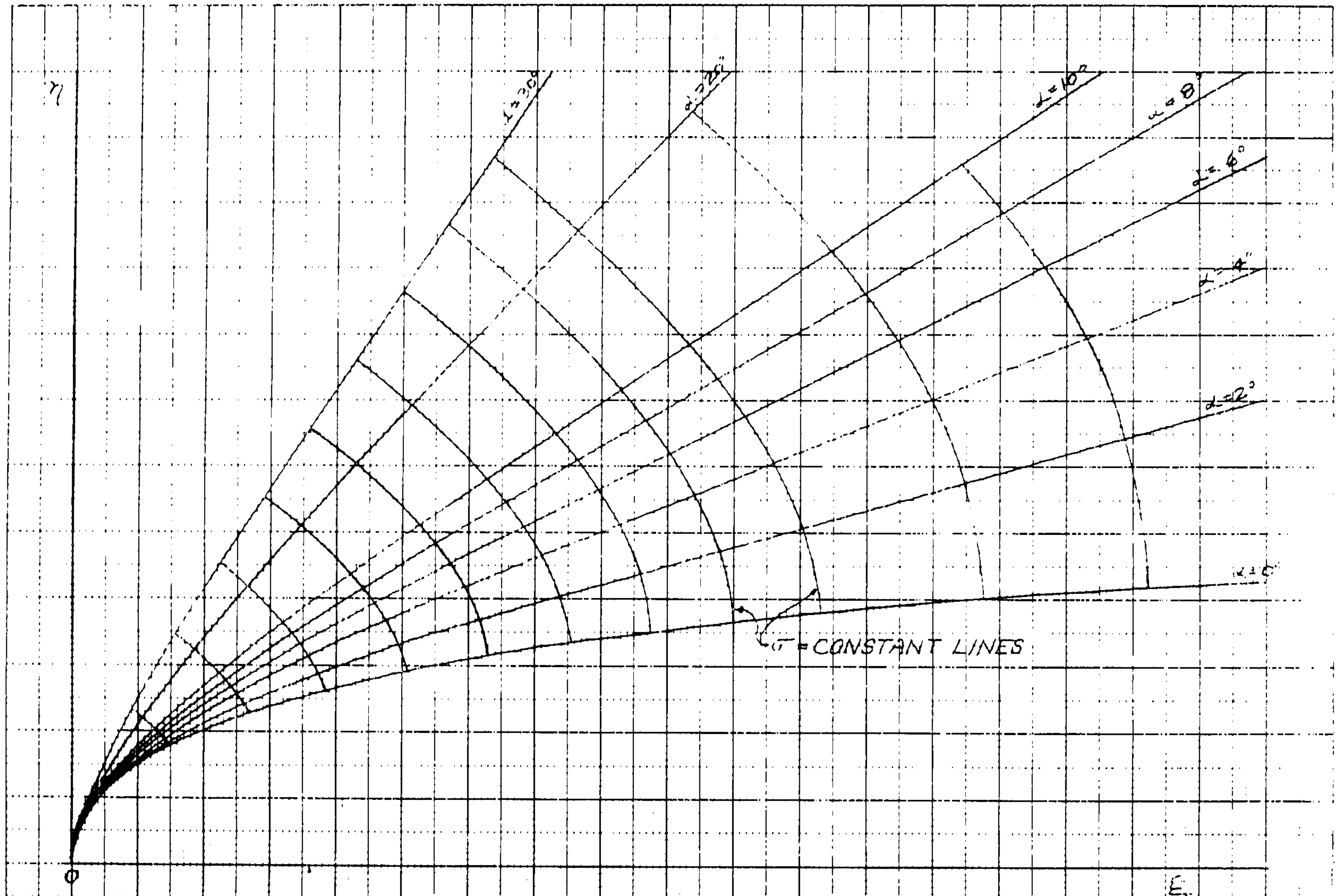


FIGURE 7. NON-DIMENSIONAL CABLE SHAPE

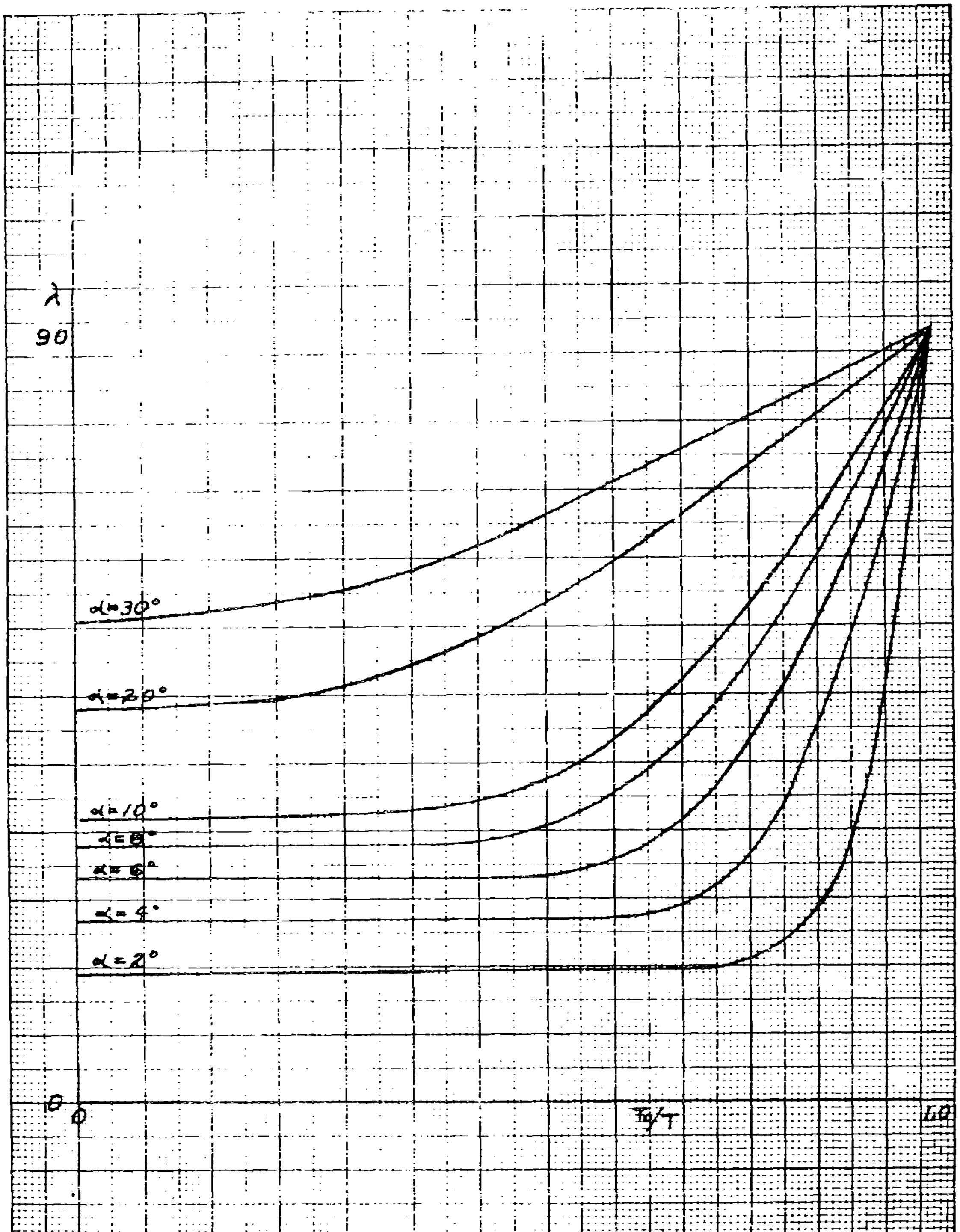


FIGURE 8. CABLE TENSION VARIATION WITH λ

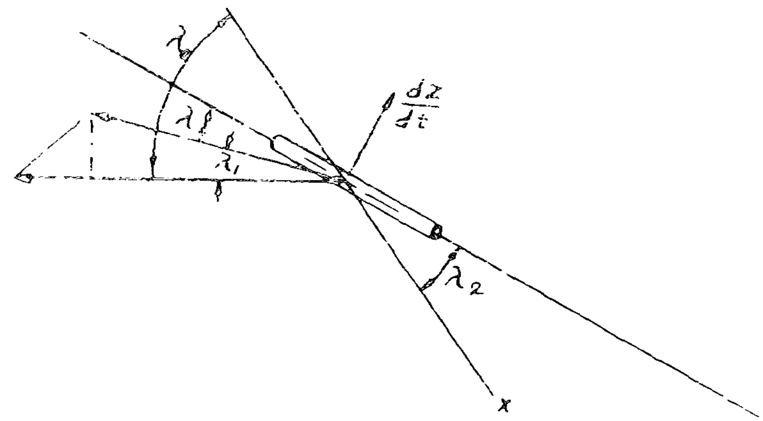


FIGURE 9. DISTURBED CABLE ELEMENT IN UNIFORM AIRSTREAM

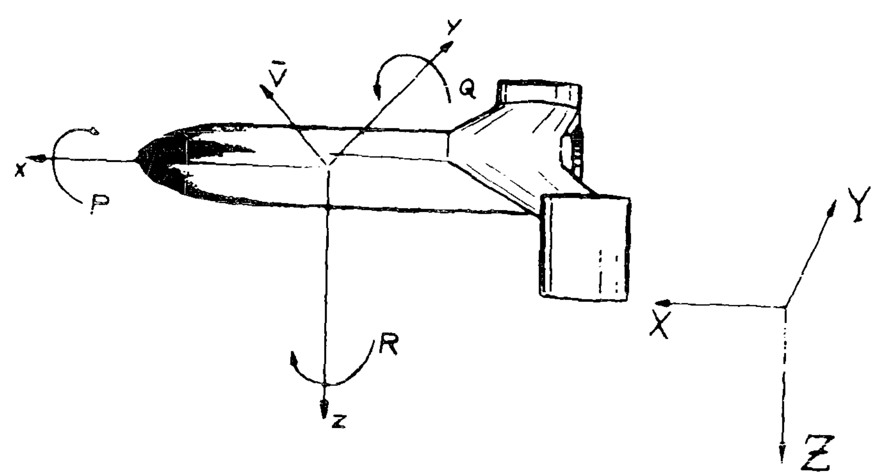


FIGURE 10. INERTIAL AXIS SYSTEMS

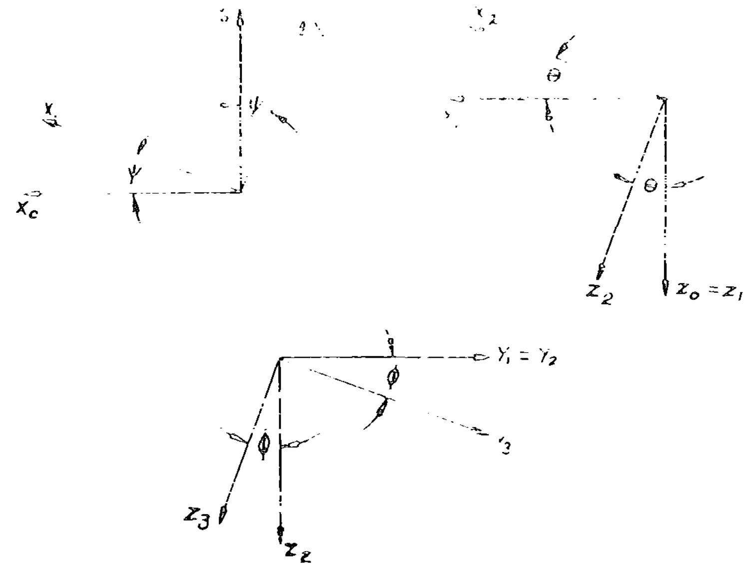


FIGURE 11. EULER ANGLE RELATIONS

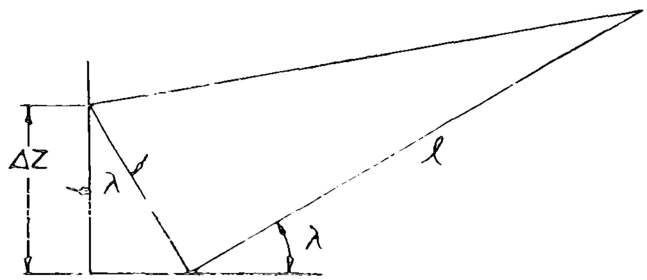
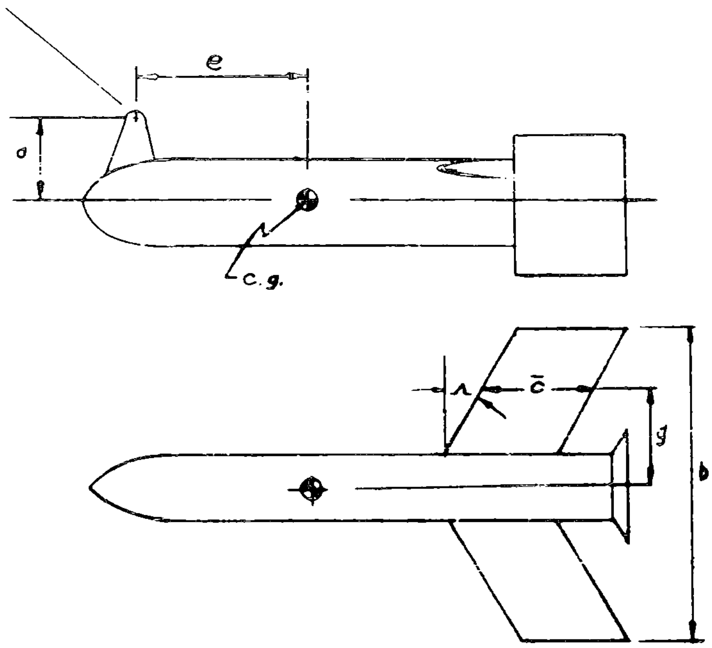


FIGURE 12. GEOMETRY - DEPENDENCE OF λ VARIATION ON LINE LENGTH



Wing area $S = 5.5\ ft^2$
Wing span $b = 4$ ft.
Mean chord $\bar{c} = 1.33$ ft.
$a = .625$ ft.
$e = 2.67$ ft.
$g = .964$ ft
Aspect Ratio $A = 2.91$
Root chord $C_R = 1.5$ ft
Tip chord $C_T = 1.25$ ft
Sweepback angle $\Lambda = 37°$

FIGURE 13. BASIC BIRD CONFIGURATION

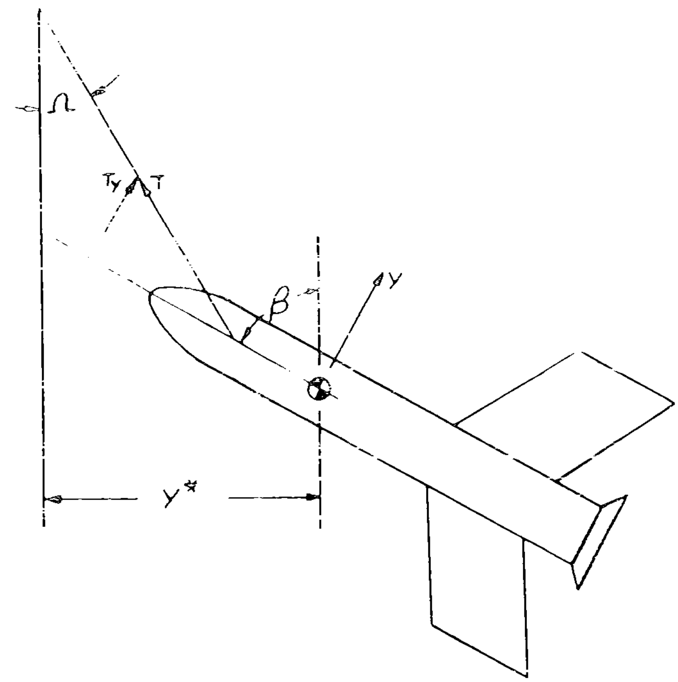


FIGURE 14. DIRECTIONAL PLANE GEOMETRY

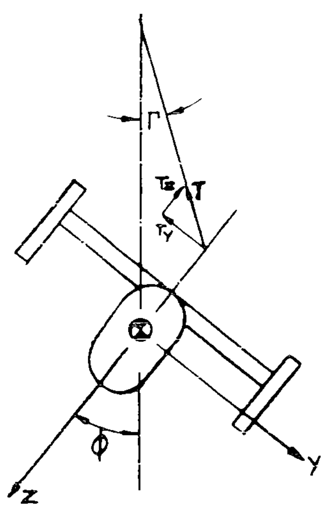


FIGURE 15. ROLL PLANE GEOMETRY

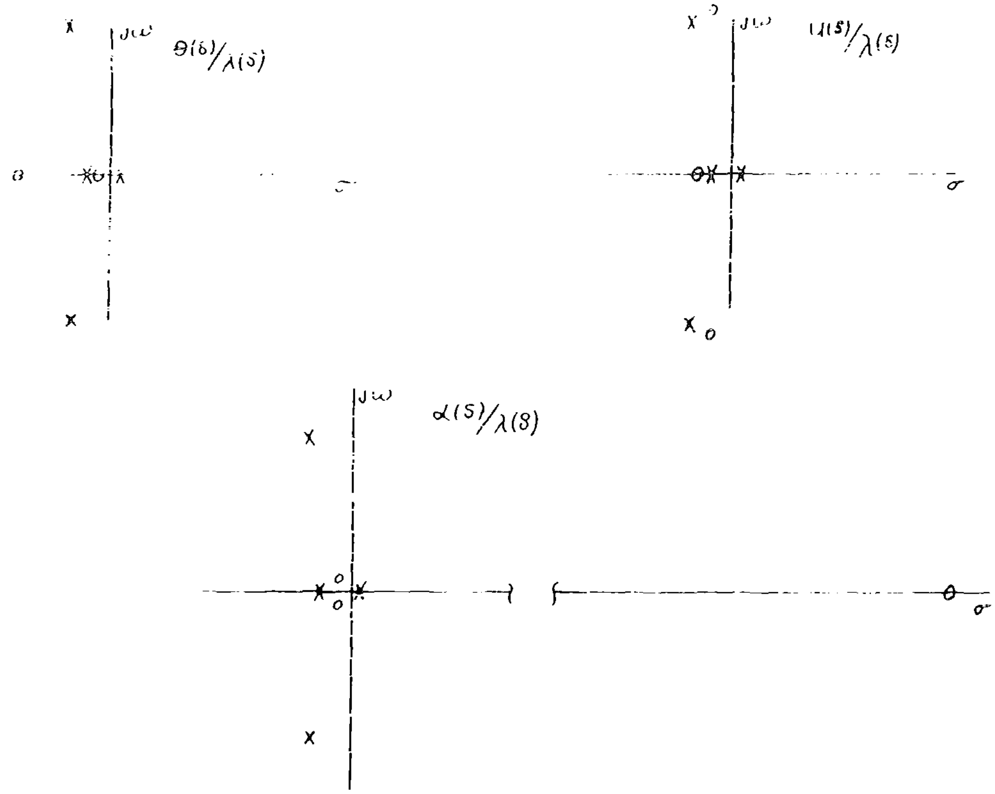


FIGURE 16. APPROXIMATE ZERO-POLE LOCATIONS FOR THE TRANSFER FUNCTIONS OF EQUATIONS IV - 14

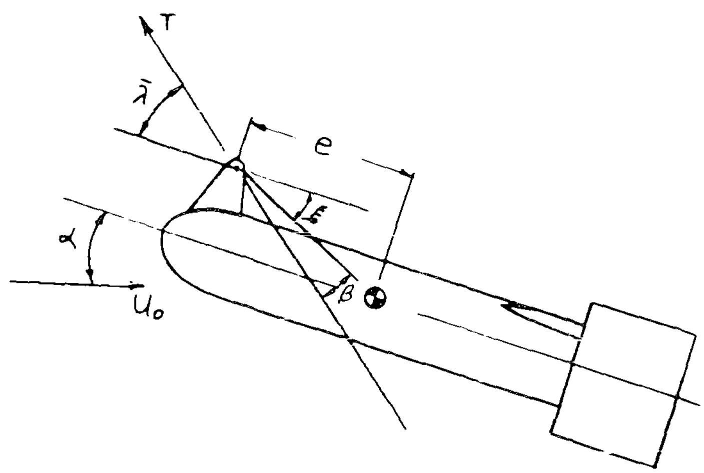


FIGURE 17. GEOMETRY FOR STATIC EFFECT ON VARIABLE TENSION

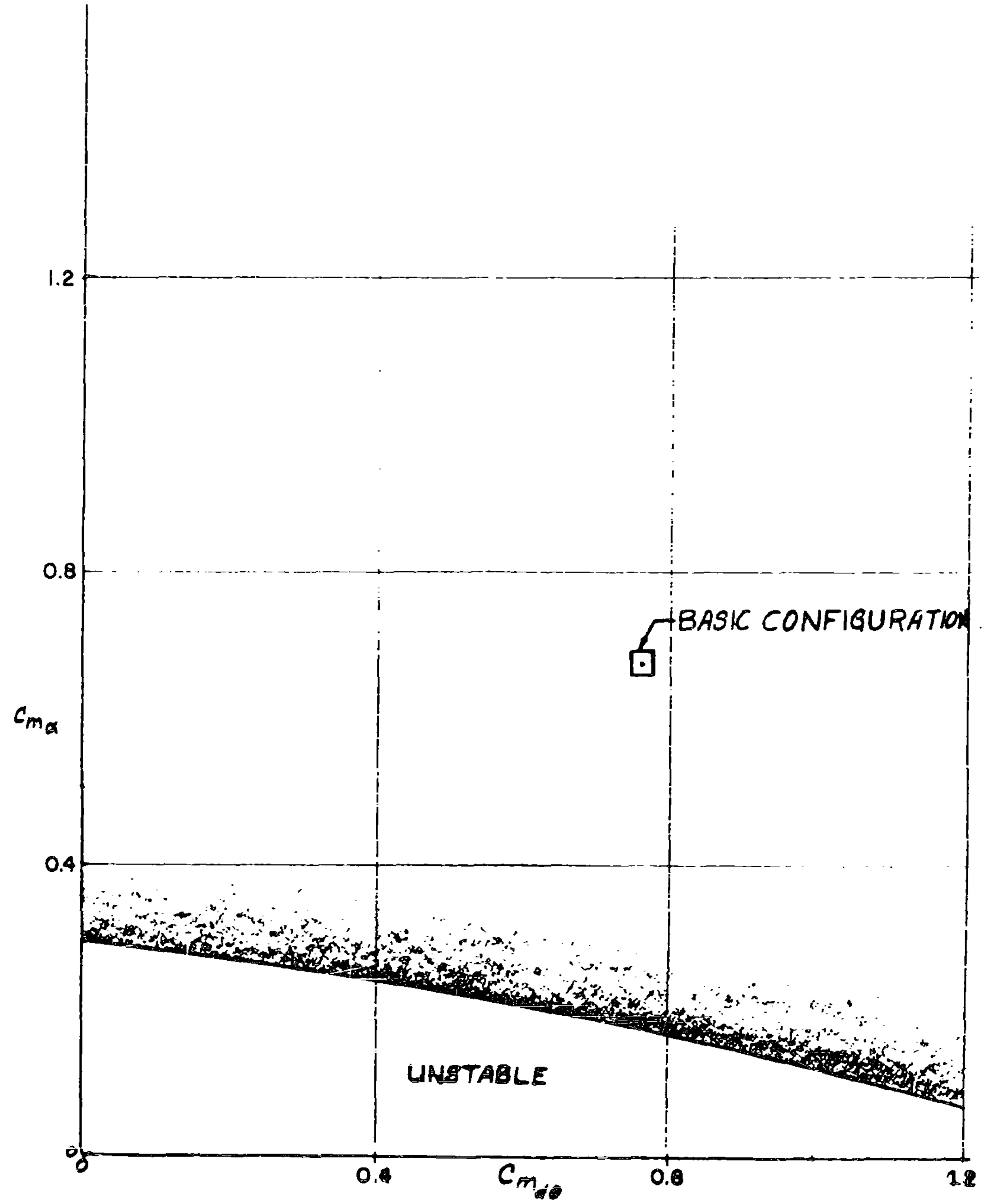


FIGURE 18. LONGITUDINAL STABILITY BOUNDARY
C_{m_α} vs. $C_{m_{\dot\theta}}$

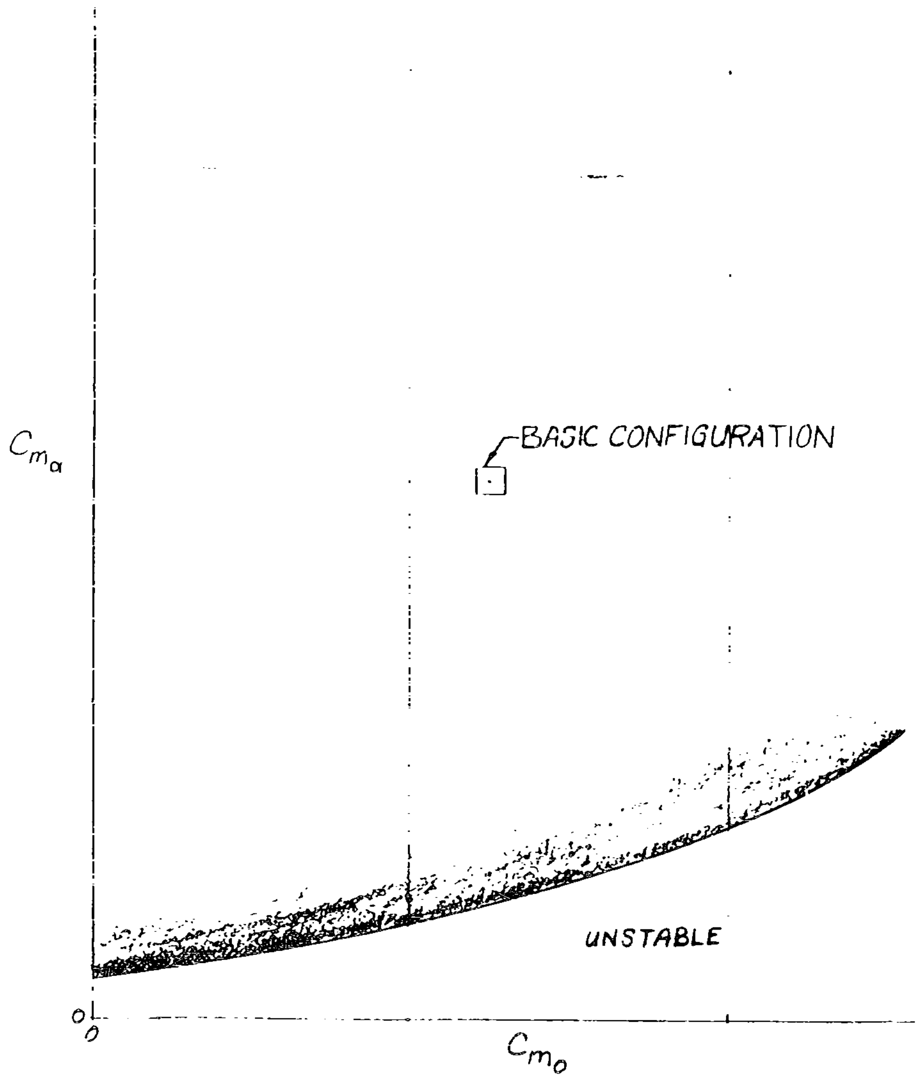


FIGURE 19. LONGITUDINAL STABILITY BOUNDARY

C_{m_α} vs. C_{m_u}

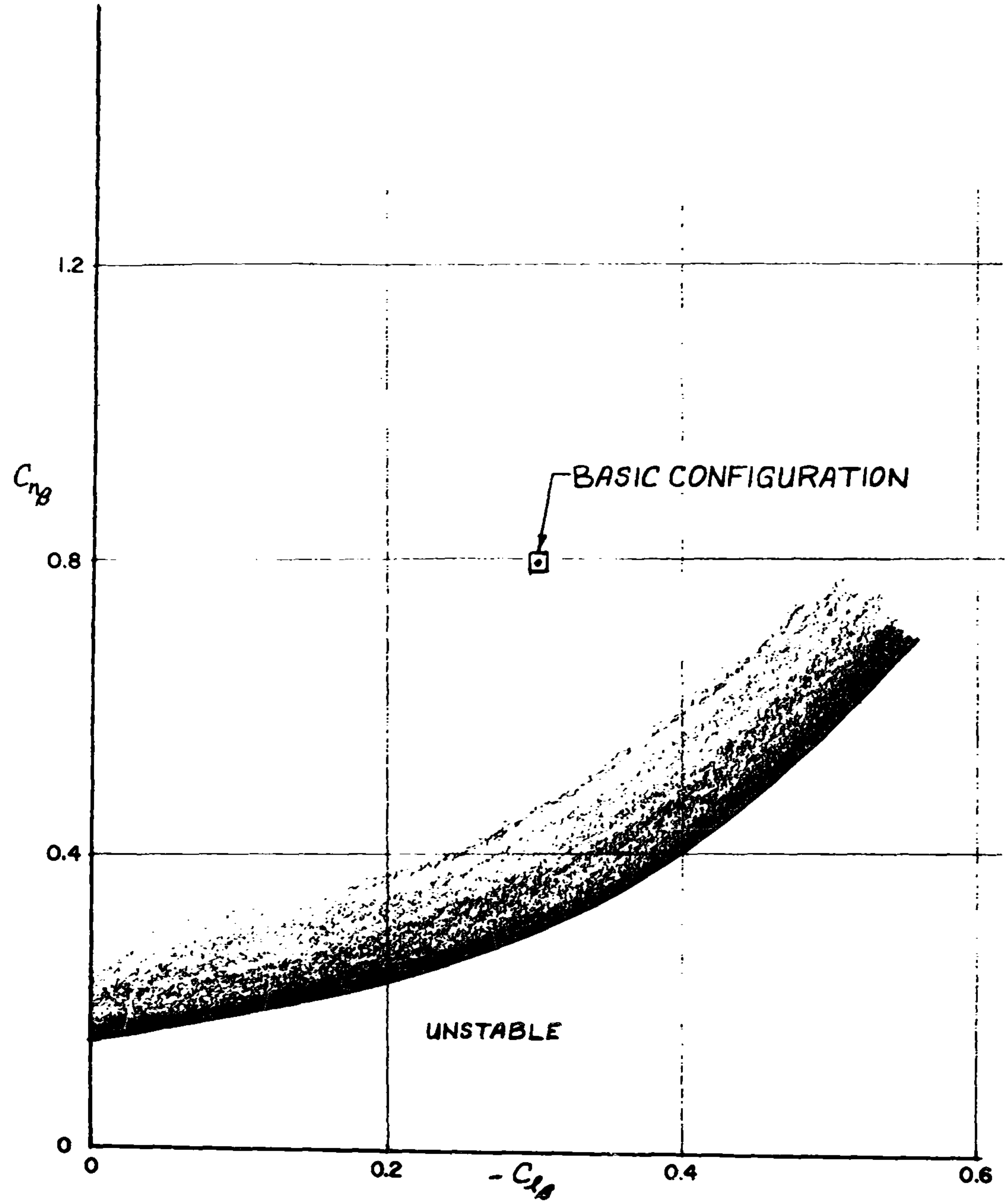


FIGURE 20. LATERAL STABILITY BOUNDARY
C_{n_β} vs. C_{ℓ_β}

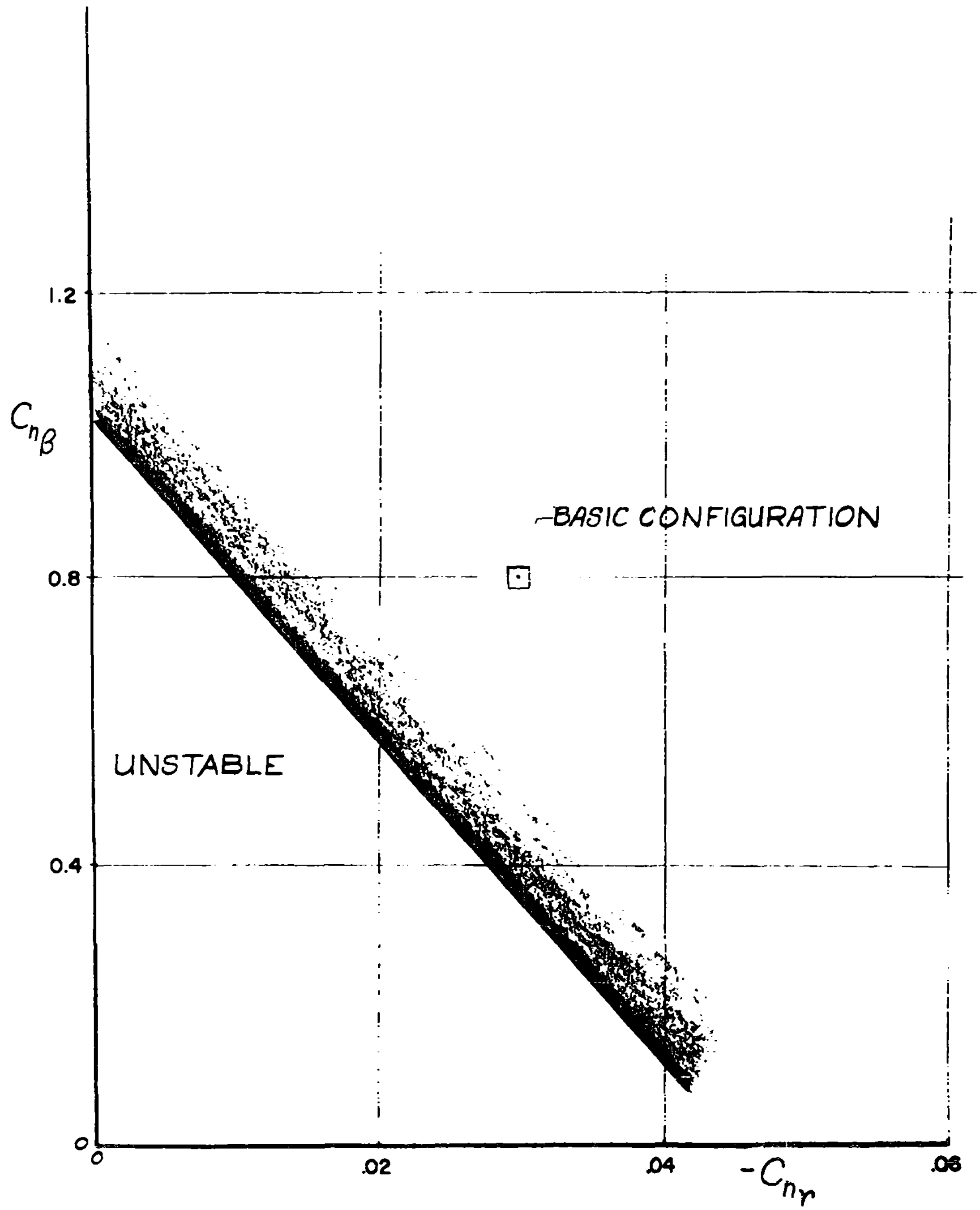


FIGURE 21. LATERAL STABILITY BOUNDARY
C_{n_β} vs. C_{n_r}

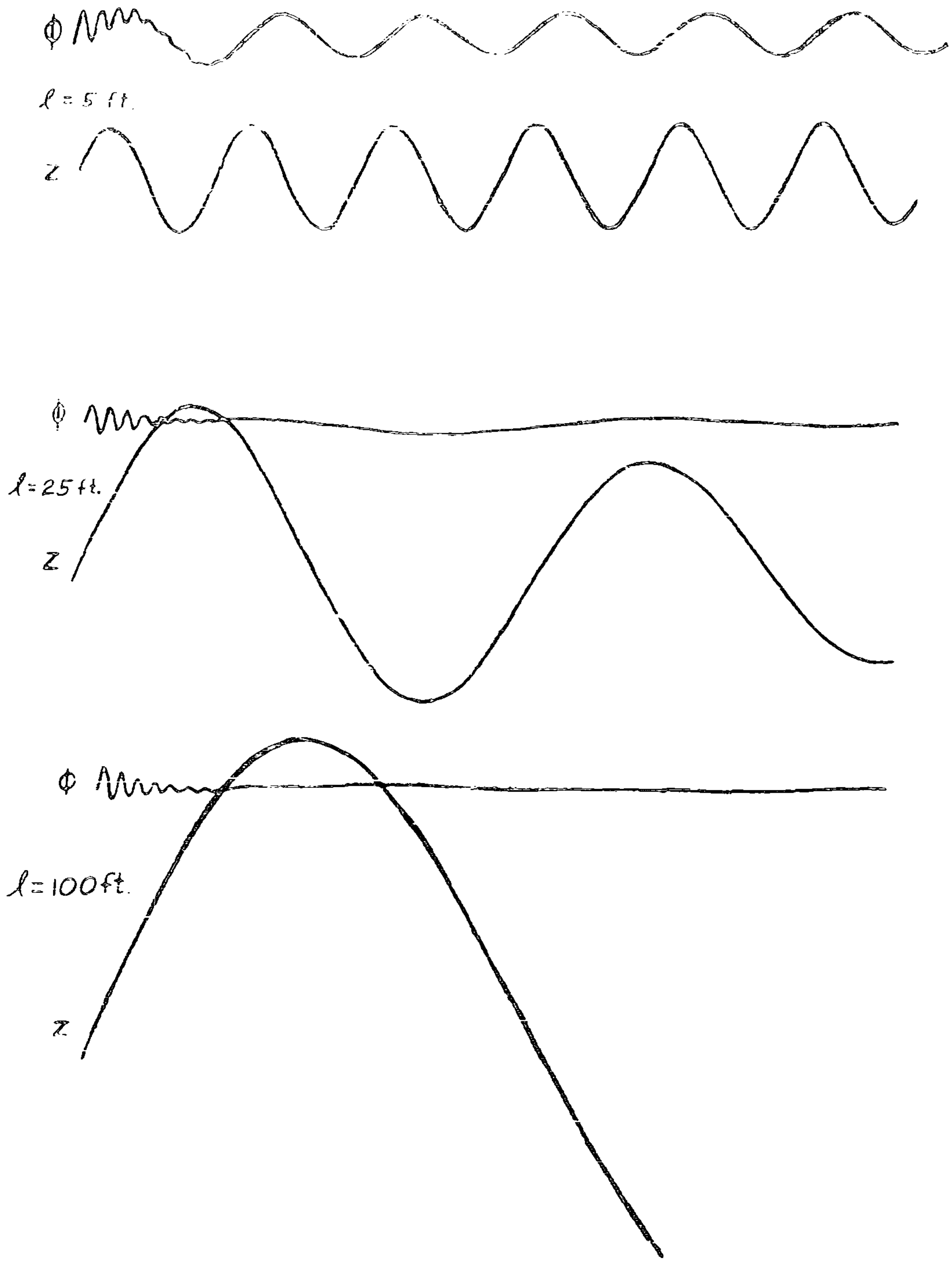


FIGURE 22. VARIATION OF PHUGOID-PENDULUM COUPLED OSCILLATIONS WITH LINE LENGTH

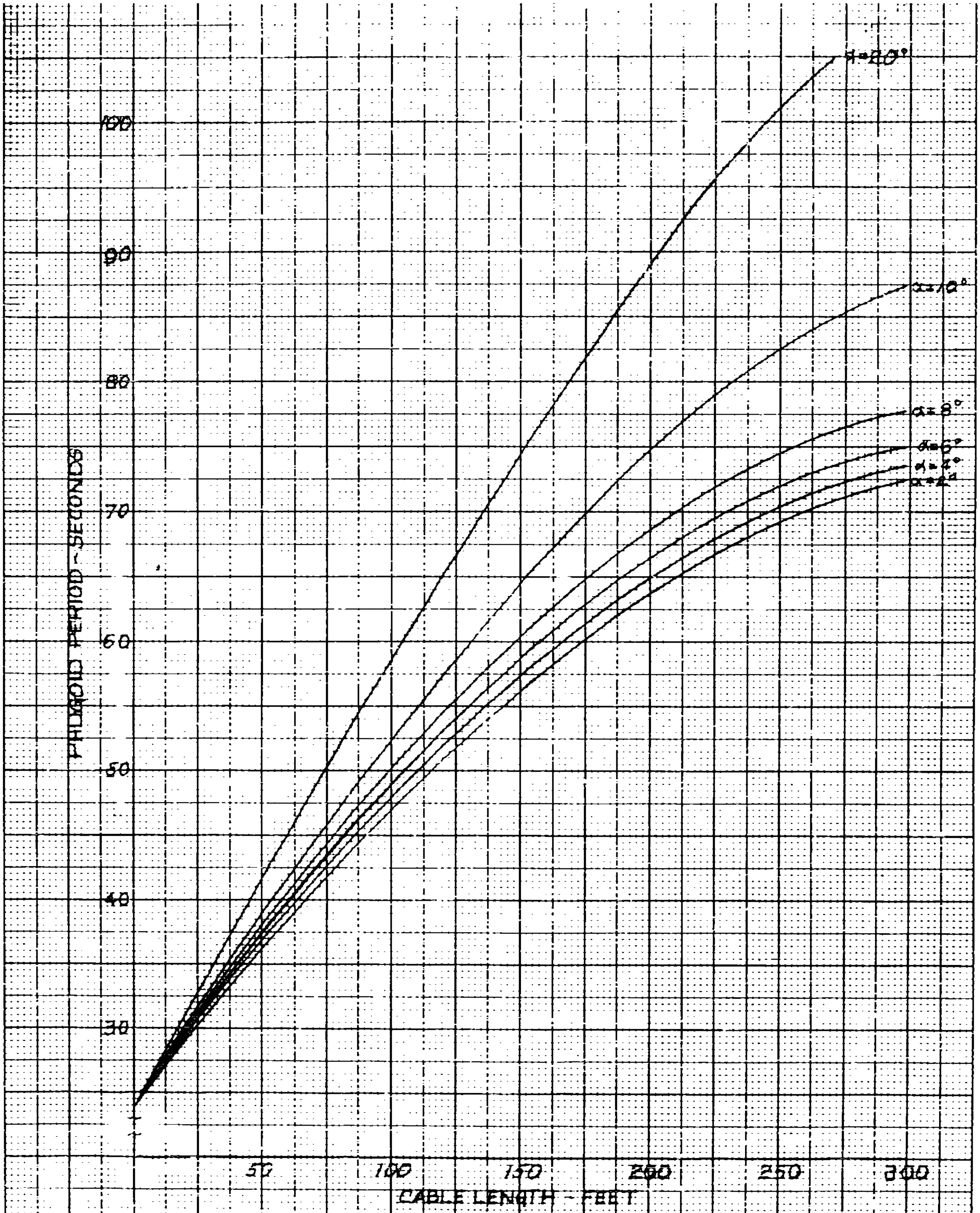


FIGURE 23. PHUGOID PERIOD VARIATION WITH CABLE LENGTH

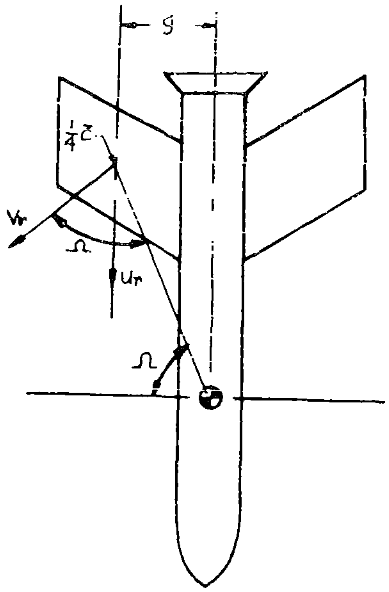


FIGURE A1. GEOMETRY FOR C_{ℓ_r} CALCULATION

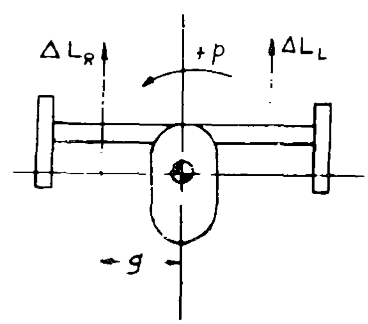


FIGURE A2. GEOMETRY FOR C_{ℓ_p} CALCULATION

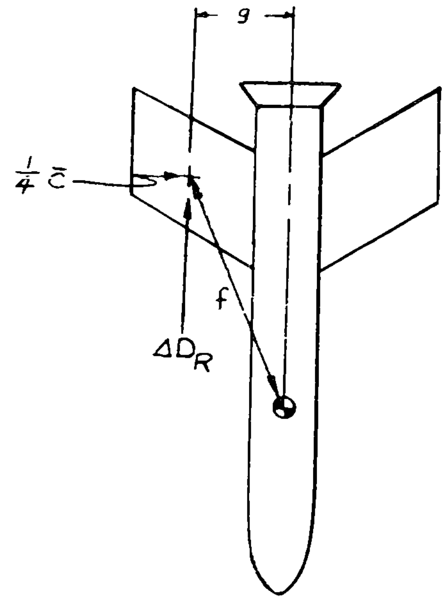


FIGURE A3. GEOMETRY FOR C_{n_p} CALCULATION

FIGURE A4. CHANGE IN DIRECTION OF LIFT VECTOR DUE TO ROLL

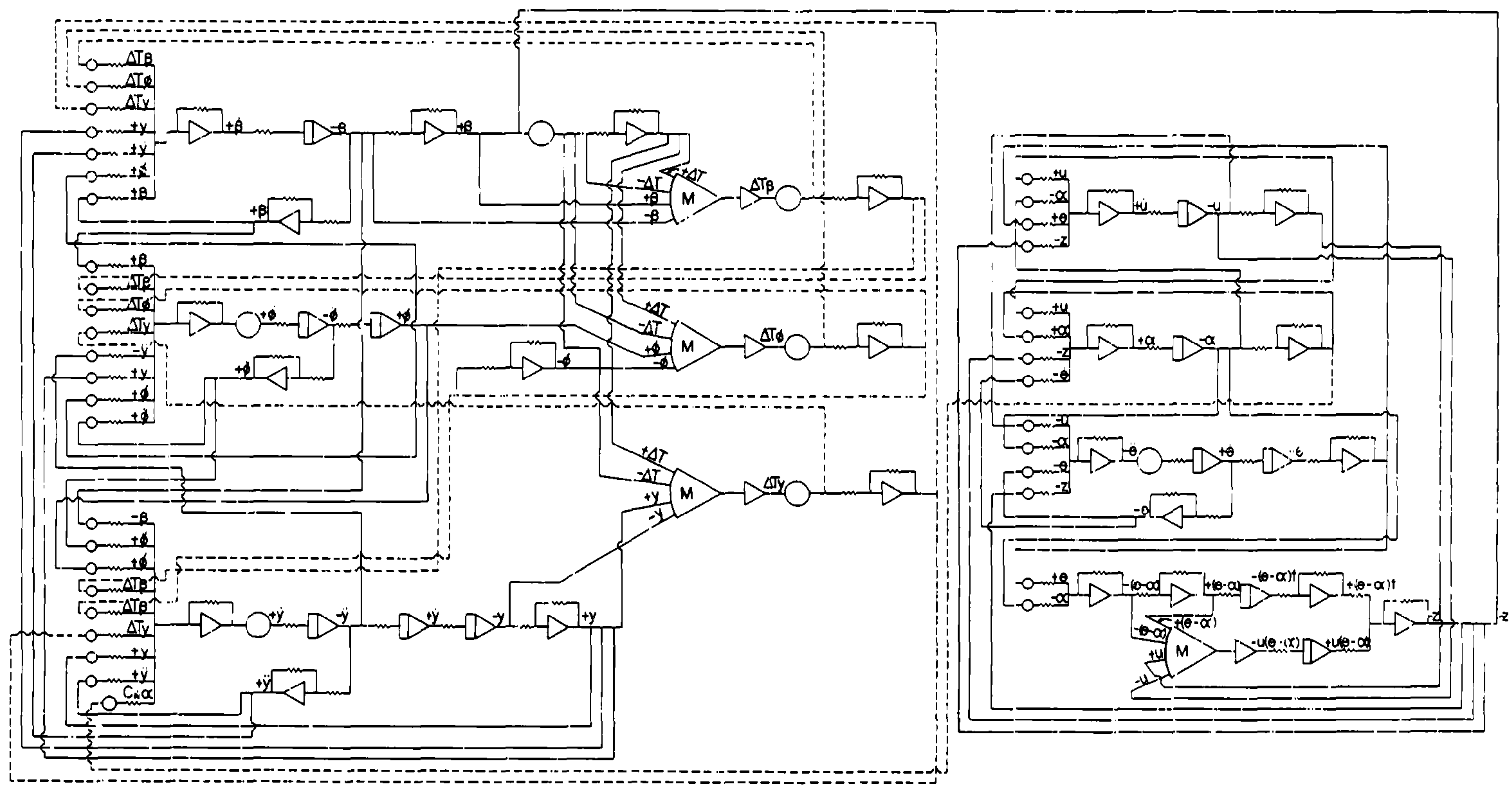


FIGURE C1. ANALOG DIAGRAM

UNCLASSIFIED

UNCLASSIFIED